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Second order linear parabolic equations in uniform spaces in $\mathbb{R}^N$ *

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Abstract

We solve second order parabolic equations with nonsmooth coefficients and initial data in suitable uniform spaces. We also show the smoothing effect of the corresponding analytic semigroup depending on the integrability properties of the coefficients. Robustness with respect to perturbations in the coefficients is also obtained.

Key words: parabolic equations, analytic semigroups, perturbation, smoothing, uniform spaces.
Mathematical Subject Classification 2010: 35B20, 35B30, 35B35, 35B65, 35K10, 35K15, 47D03, 47D06.

1 Introduction

In this paper we address the solvability of some second order linear parabolic equations in $\mathbb{R}^N$. In particular, we study the problem

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 \qquad x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) \qquad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where the coefficients of the elliptic principal part of the equation are assumed to be bounded and uniformly continuous, that is, $a_{kl} \in BUC(\mathbb{R}^N)$. The lower order coefficients are assumed to have some local integrability properties and no asymptotic decay as $|x| \to \infty$ whatsoever. More precisely they are assumed to belong to some locally uniform Lebesgue spaces. To be more precise, let $L^p_U(\mathbb{R}^N)$ denote the locally uniform space composed of the functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |f|^p \leq C \tag{1.2}$$

endowed with the norm

$$\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}$$

(for $p = \infty$, $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$). Also define $\dot{L}^p_U(\mathbb{R}^N)$ as the closed subspace of $L^p_U(\mathbb{R}^N)$ consisting in elements which are translation continuous with respect to $\|\cdot\|_{L^p_U(\mathbb{R}^N)}$.

On the other hand, the initial data in (1.1) will also assumed to belong to some uniform Bessel space $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$; see Section 2 for further details. Our goal in this paper is to prove the following

Theorem 1.1 *Assume for $j = 1, \dots, N$,*

$$\|b_j\|_{\dot{L}^{p_j}_U(\mathbb{R}^N)} \leq R_j \quad \textit{and} \quad \|c\|_{\dot{L}^{p_0}_U(\mathbb{R}^N)} \leq R_0$$

where $p_j > N$ and $p_0 > \frac{N}{2}$. Define $a_0 = 0$, $a_j = 1$ and for $j = 1, \dots, N$ and, $\tilde{p} = \min\{p_j,\ j = 1, \dots, N\} > N$. If $q' < \tilde{p}$ and $q > p_0$, we will also assume $p_0 > \frac{Nq}{N+q}$.

Then for any $1 < q < \infty$ there exists non-empty interval $I(q) \subset (-\frac{1}{2}, 1)$ containing $(-1 + \max_j\{\frac{a_j}{2} + \frac{N}{2p_j}\}, 1 - \max_j\{\frac{N}{2p_j}\})$, such that for any $\gamma \in I(q)$, we have a strongly continuous, order preserving, analytic semigroup $S(t)$ in the space $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$, for the problem

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{1.3}$$

with $u(t;u_0) = S(t)u_0$, $t \geq 0$.

Moreover the semigroup has the smoothing estimate

$$\|S(t)u_0\|_{\dot{H}^{2\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{2\gamma}_U(\mathbb{R}^N) \tag{1.4}$$

for every $\gamma, \gamma' \in I(q)$ with $\gamma' \geq \gamma$, and

$$\|S(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N) \tag{1.5}$$

for $1 < q \leq r \leq \infty$ with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on R_j and R_0.

Furthermore,

$$I(q) = (-1 + \max_{j=0,\dots,N}\{\frac{a_j}{2} + \frac{N}{2}(\frac{1}{p_j} - \frac{1}{q})_+\}, 1 - \frac{N}{2}(\frac{1}{\min_{j=0,\dots,N}\{p_j\}} - \frac{1}{q})_+). \tag{1.6}$$

Finally, if, as $\varepsilon \to 0$

$$b_j^\varepsilon \to b_j \quad in\ \dot{L}_U^{p_j}(\mathbb{R}^N), \quad p_j > N, \quad j = 1, \dots, N,$$

$$c^\varepsilon \to c \quad in\ \dot{L}_U^{p_0}(\mathbb{R}^N), \quad p_0 > N/2$$

then for every $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(\dot{H}_U^{2\gamma,q}(\mathbb{R}^N), \dot{H}_U^{2\gamma',q}(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \le T$$

for all $\gamma, \gamma' \in I(q)$*,* $\gamma' \ge \gamma$ *and for all* $1 < q \le r \le \infty$*,*

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(\dot{L}_U^q(\mathbb{R}^N), \dot{L}_U^r(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \le T.$$

A similar result, without the continuity with respect of perturbations in the coefficients, was proved in Theorem 5.3 in [3], assuming additionally that

$$p_j \ge q > 1, \quad \text{for } j = 0, \dots, N.$$

That result was later recovered in [5] with different techniques. The result in [3, 5] just allowed for $\gamma \ge 0$ in (1.4). Here, we remove such restrictions allowing in particular, a larger class of initial data, since in (1.4), γ can be even negative. Also, with the additional assumptions above, Theorem 1.1 recovers Theorem 5.3 in [3].

Note that the additional assumption in Theorem 1.1 that "if $q' < \tilde{p}$ and $q > p_0$, we will also assume $p_0 > \frac{Nq}{N+q}$" applies only when q is large relative to the exponents p_j, $j = 0, \dots, N$. For $1 < q \le N$ this imposes no additional restriction since in this range $\frac{Nq}{N+q} \le \frac{N}{2}$. Furthermore, since $\frac{Nq}{N+q} < N$ for all q, if $p_0 \ge N$ no additional assumption is imposed either.

It is also worth mentioning that the estimates (1.4), (1.5) on the semigroups of solutions of (1.3) are uniform with respect to bounded famillies of coefficients. Finally Theorem 1.1 gives the continuity of the semigroups with respect to perturbations in the lower order coefficients of (1.3).

The paper is organized as follows. In Section 2 we briefly recall the main properties of uniform Lebesgue and Bessel spaces that will be needed hereafter. Suitable references are given for the interested reader. Then in Section 3 we prove Theorem 1.1. For this we first consider the case in which $b_j = 0$ and $c_0 = 0$, for which we use the results in [3], [2] and [1]. Then we consider (1.1) as a perturbation of the previous case and apply the techniques in [5].

Along similar lines, fourth order parabolic problems have been analyzed in [4].

2 Some properties of uniform Bessel spaces

Consider the locally uniform space $L^q_U(\mathbb{R}^N)$ for $1 \leq q \leq \infty$ defined as in (1.2) and denote by $\dot{L}^q_U(\mathbb{R}^N)$ the closed subspace of $L^q_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{L^q_U(\mathbb{R}^N)}$. That is

$$\|\tau_y\phi - \phi\|_{L^q_U(\mathbb{R}^N)} \to 0 \text{ as } |y| \to 0,$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations in $\mathbb{R}^N$. With this we get $L^q(\mathbb{R}^N) \subset \dot{L}^q_U(\mathbb{R}^N)$ for $1 \leq q < \infty$ and for $q = \infty$ we get $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ and $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$.

Thus we introduce the *uniform Bessel-Sobolev spaces* $H^{k,q}_U(\mathbb{R}^N)$, with $k \in \mathbb{N}$, as the set of functions $\phi \in H^{k,q}_{loc}(\mathbb{R}^N)$ such that

$$\|\phi\|_{H^{k,q}_U(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{H^{k,q}(B(x,1))} < \infty$$

for $k \in \mathbb{N}$, where $H^{k,q}(B(x,1))$ is the standard Bessel space. Then denote by $\dot{H}^{k,q}_U(\mathbb{R}^N)$ a subspace of $H^{k,q}_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{H^{k,q}_U(\mathbb{R}^N)}$, that is

$$\|\tau_y\phi - \phi\|_{H^{k,q}_U(\mathbb{R}^N)} \to 0 \text{ as } |y| \to 0$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations.

To construct intermediate spaces of noninteger order, consider the complex interpolation functor denoted by $[\cdot,\cdot]_\theta$, for $\theta \in (0,1)$, see [6] for details. Then for $1 \leq q < \infty$, $k \in \mathbb{N} \cup \{0\}$ and $s \in (k, k+1)$ we define $\theta \in (0,1)$ such that $s = \theta(1+k) + (1-\theta)k$, that is $\theta = s - k$. Then one can define the intermediate spaces by interpolation as

$$H^{s,q}_U(\mathbb{R}^N) = [H^{k+1,q}_U(\mathbb{R}^N), H^{k,q}_U(\mathbb{R}^N)]_\theta, \tag{2.1}$$

and

$$\dot{H}^{s,q}_U(\mathbb{R}^N) = [\dot{H}^{k+1,q}_U(\mathbb{R}^N), \dot{H}^{k,q}_U(\mathbb{R}^N)]_\theta. \tag{2.2}$$

For details on this construction, see [2, 3].

Using Proposition 4.2 in [3] it is easy to see that the sharp embeddings of Bessel spaces translate into

$$\dot{H}^{s,q}_U(\mathbb{R}^N) \subset \begin{cases} \dot{L}^r_U(\mathbb{R}^N),\ s - \frac{N}{q} \geq -\frac{N}{r},\ 1 \leq r < \infty & if\ s - \frac{N}{q} < 0 \\ \dot{L}^r_U(\mathbb{R}^N),\ 1 \leq r < \infty & if\ s - \frac{N}{q} = 0 \\ C^\eta_b(\mathbb{R}^N) & if\ s - \frac{N}{q} > \eta \geq 0. \end{cases} \tag{2.3}$$

In [3], the Laplace operator was considered in the scale of spaces $H^{s,q}_U(\mathbb{R}^N)$ and $\dot{H}^{s,q}_U(\mathbb{R}^N)$, $s \geq 0$, and it was proved that $-\Delta$ defines an analytic semigroup. However in the "undotted" spaces the semigroup generated by $-\Delta$ is analytic but not strongly continuous. These spaces are less convenient to use because smooth functions are not dense in them; see [3]. It was moreover proved in [3, Theorem 5.3, pg. 290], that for some $\mu \in \mathbb{R}$, $-\Delta + \mu I$ has bounded imaginary powers in $\dot{L}^q_U(\mathbb{R}^N)$, and therefore the interpolation spaces $\dot{H}^{s,q}_U(\mathbb{R}^N)$, $s \geq 0$, coincide with the fractional powers ones; see [2, V.1.5.13, pg. 283]. Also, note that from the results in [3] we have in particular that $\dot{H}^{1,q}_U(\mathbb{R}^N) = [\dot{H}^{2,q}_U(\mathbb{R}^N), \dot{L}^q_U(\mathbb{R}^N)]_{1/2}$; see Remark 5.7, page 291 in that reference. From this reiterations properties of interpolation gives that $\dot{H}^{2\theta,q}_U(\mathbb{R}^N) = [\dot{H}^{2,q}_U(\mathbb{R}^N), \dot{L}^q_U(\mathbb{R}^N)]_\theta$ for $\theta \in [0,1]$.

The uniform spaces above can be extended to negative indexes by a general extrapolation procedure as in [2]. In this way one can define the extrapolated space $\dot{H}^{-k,q}_U(\mathbb{R}^N)$ as the completion of $\dot{L}^q_U(\mathbb{R}^N)$ with the norm $\|(-\Delta + I)^{-k/2}u\|_{\dot{L}^q_U(\mathbb{R}^N)}$. Again, by complex interpolation, for $0 < s < k$, $k \in \mathbb{N}$, the intermediate spaces are given by

$$\dot{H}^{-s,q}_U(\mathbb{R}^N) = [\dot{L}^q_U(\mathbb{R}^N), \dot{H}^{-k,q}_U(\mathbb{R}^N)]_\theta, \qquad \text{with } \theta = \frac{s}{k}.$$

Note that because of the reiteration property of the complex interpolation (see (2.8.4) in page 31 in [2] and Theorem 1.5.4 in [2]) this definition of $\dot{H}^{-s,q}_U(\mathbb{R}^N)$ does not depend on k.

For the standard (not uniform) Bessel spaces, there is a simple characterization for the spaces with negative indexes using duality and reflexivity, see [2, V.1.5.12, pg. 282]. However, since the uniform spaces are not reflexive, even for $q = 2$, there is no simple characterization of the uniform spaces with negative indexes. However the following result which was proved in [4, Proposition 3.1] turns out to be very useful in what follows. Note that this result is formaly the one we would expect by duality if the spaces were reflexive.

Lemma 2.1 *The following embedding holds*

$$\dot{L}^p_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{-s,q}_U(\mathbb{R}^N), \qquad \text{if} \quad s - \frac{N}{q'} \geq -\frac{N}{p'}, \qquad s > 0.$$

3 Parabolic equations in uniform spaces

In order to prove Theorem 1.1 we first recall some results from Section 5 in [3]. Consider the operator

$$A_0 := -\sum_{k,l=1}^{N} a_{kl}(x)\partial_k\partial_l$$

where we assume $a_{kl} \in BUC(\mathbb{R}^N)$. Hence, for some modulus of continuity ω, we have the norm

$$\|a_{kj}\|_{BUC(\mathbb{R}^N,\omega)} = \sup_{x\in\mathbb{R}^N} |a_{kj}(x)| + \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|a_{kj}(x) - a_{kj}(y)|}{\omega(|x-y|)}. \tag{3.1}$$

We also assume the following ellipticity condition: for some constants $M > 0$ and $\theta_0 \in (0, \frac{\pi}{2})$, the following holds

$$A_0(x,\xi) \geq \frac{1}{M} > 0, \ \ |arg(A_0(x,\xi))| \leq \theta_0, \text{ for all } x, \xi \in \mathbb{R}^N \text{ with } |\xi| = 1. \tag{3.2}$$

Note that M can be chosen such that $\|a_{kj}\|_{BUC(\mathbb{R}^N,\omega)} < M$. Finally, we will assume

$$\int_0^1 \frac{\omega^{1/3}(t)}{t}\, dt < \infty. \tag{3.3}$$

Note that these assumptions are satisfied for the case $a_{kl} = \delta_{kl}$, i.e. when $A_0 = -\Delta$.

Under these assumptions we get the following.

Proposition 3.1 *Under the above assumptions, for any $1 < q < \infty$ and $\beta \in \mathbb{R}$, the problem*

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u = 0, & x \in \mathbb{R}^N, \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{3.4}$$

where $u_0 \in \dot{H}_U^{2\beta,q}(\mathbb{R}^N)$, has a unique solution $u(t;u_0)$ that satisfies the smoothing estimates

$$\|u(t;u_0)\|_{\dot{H}_U^{2\alpha,q}(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}_U^{2\beta,q}(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{H}_U^{2\beta,q}(\mathbb{R}^N) \tag{3.5}$$

for any $\alpha \geq \beta$, with some $\mu_0 > 0$ which depends only on q, M and θ_0.

In particular, setting $u(t;u_0) = S_0(t)u_0$ for $t \geq 0$, defines an order preserving, C^0 analytic semigroup $S_0(t)$ in $\dot{H}_U^{2\beta,q}(\mathbb{R}^N)$.

Proof. Because of Theorem 5.3 in [3], for any $1 < q < \infty$, A_0 generates an analytic semigroup in $\dot{L}_U^q(\mathbb{R}^N)$ with domain $\dot{H}_U^{2,q}(\mathbb{R}^N)$ which, for some $\mu \in \mathbb{R}$, $A_0 - \mu I$ has bounded imaginary powers. Thus the complex interpolation spaces (2.2) coincide with the fractionary power ones; see Theorem V.1.5.13, pg. 283 in [2].

Then as a consequence of the techniques in chapter V.1 in [2] we have that in fact a suitable extension of A_0 generates an analytic semigroup in $\dot{H}_U^{\beta,q}(\mathbb{R}^N)$ with domain $\dot{H}_U^{\beta+1,q}(\mathbb{R}^N)$ and the solutions of (3.4) satisfy (3.5). Hence, $S_0(t)$ is a C^0 analytic semigroup in $\dot{H}_U^{\beta,q}(\mathbb{R}^N)$.

The fact that μ_0 depends only on q, M and θ_0, follows from [3] and the results in [2] quoted above.

For the order preserving property, recall from [1] that for coefficients $a_{kl}(x)$ as above and regular initial data, if $u_0 \geq 0$ then $S_0(t)u_0 \geq 0$ for all $t \geq 0$. Now, for $u_0 \in \dot{H}_U^{\beta,q}(\mathbb{R}^N)$ take $\{u_0^n\}_{n\in\mathbb{N}}$ regular such that $u_0^n \to u_0$ then $S_0(t)u_0^n \to S_0(t)u_0$ and since $S_0(t)u_0^n \geq 0$ for all $n \in \mathbb{N}$ then $S_0(t)u_0 \geq 0$. Note that this can be done because we are using the "dotted" spaces, where regular functions are dense. ■

To use this result for A_0 we consider (1.1) as a perturbation of (3.4). First, from Lemma 26 in [5], we have

Lemma 3.2 *i) Assume that $m \in L_U^p(\mathbb{R}^N)$, then the multiplication operator*

$$Pu(x) = m(x)u(x)$$

satisfies, for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(L_U^r(\mathbb{R}^N), L_U^s(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(L_U^r(\mathbb{R}^N),L_U^s(\mathbb{R}^N))} \leq C\|m\|_{L_U^p(\mathbb{R}^N)}.$$

ii) If moreover $m \in \dot{L}^p_U(\mathbb{R}^N)$ we have for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N))} \leq C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

Combinig this with Lemma 2.1 we get, denoting by $(x)_-$ the negative part of a real number x, and by D^a any derivative of order $a \in \mathbb{N}$, the following result.

Proposition 3.3 *Let $Pu = d(x)D^a u$, with $d \in \dot{L}^p_U(\mathbb{R}^N)$, $a \in \mathbb{N}$ and let $s \geq a$, $\sigma \geq 0$. Then for $1 < q < \infty$, if*

$$(s - a - \frac{N}{q})_- + (\sigma - \frac{N}{q'})_- \geq -\frac{N}{p'} \tag{3.6}$$

we have

$$P \in \mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N))} \leq C\|d\|_{\dot{L}^p_U(\mathbb{R}^N)}.$$

Proof. First note that $u \in \dot{H}^{s,q}_U(\mathbb{R}^N)$, thus $D^a u \in \dot{H}^{s-a,q}_U(\mathbb{R}^N)$. Because of (3.6) we can choose $r, \rho \geq 1$ such that $(s - a - \frac{N}{q})_- \geq -\frac{N}{r}$ and $(\sigma - \frac{N}{q'})_- \geq -\frac{N}{\rho'}$ with $\frac{1}{\rho} = \frac{1}{r} + \frac{1}{p}$ (and so $r \geq p'$).

Therefore we can use the inclusion $\dot{H}^{s-a,q}_U(\mathbb{R}^N) \hookrightarrow \dot{L}^r_U(\mathbb{R}^N)$, see (2.3), and then because of Lemma 3.2 we get $P_a u \in \dot{L}^\rho_U(\mathbb{R}^N)$. Now we use the inclusion $\dot{L}^\rho_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)$ from Lemma 2.1 and we get the result. ■

Thus we get the following result for second order operators, where A_0 is perturbed by some lower order term.

Theorem 3.4 *Let $a \in \{0, 1\}$, $d \in \dot{L}^p_U(\mathbb{R}^N)$ be such that $\|d\|_{\dot{L}^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{2-a}$. Then for any $1 < q < \infty$ and any P as above there exists an interval $I(q, a) \subset (-1 + \frac{a}{2}, 1)$ containing $(-1 + \frac{a}{2} + \frac{N}{2p}, 1 - \frac{N}{2p})$, such that for any $\gamma \in I(q, a)$, we have an order preserving, stronlgy continuous, analytic semigroup $S_P(t)$ in the space $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + A_0 u + d(x)D^a u = 0, & x \in \mathbb{R}^N, \ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.7}$$

with $u(t; u_0) = S_P(t)u_0$, $t \geq 0$.

Moreover the semigroup has the smoothing estimate

$$\|S_P(t)u_0\|_{\dot{H}^{2\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma' - \gamma}} \|u_0\|_{\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0, \ u_0 \in \dot{H}^{2\gamma}_U(\mathbb{R}^N) \tag{3.8}$$

for every $\gamma, \gamma' \in I(q, a)$ with $\gamma' \geq \gamma$, and

$$\|S_P(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0, \ u_0 \in \dot{L}^q_U(\mathbb{R}^N) \tag{3.9}$$

for $1 < q \leq r \leq \infty$ with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on M, θ_0 and R_0.

The interval $I(q,a)$ is given by

$$I(q,a) = (-1 + \frac{a}{2} + \frac{N}{2}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{N}{2}(\frac{1}{p} - \frac{1}{q})_+) \subset (-1 + \frac{a}{2}, 1).$$

Finally, if, as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad in\ \dot{L}^p_U(\mathbb{R}^N), \quad p > \frac{N}{2-a}$$

then for every $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{H}^{2\gamma,q}_U(\mathbb{R}^N), \dot{H}^{2\gamma',q}_U(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \le T$$

for all $\gamma, \gamma' \in I(q,a)$, $\gamma' \ge \gamma$ and for all $1 < q \le r \le \infty$,

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^r_U(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \le T.$$

Proof. We first prove (3.8) and for this we follow several steps.
Step 1. Denote $X_\alpha := \dot{H}^{2\alpha,q}_U(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$. If we assume for a moment that (3.6) is satisfied for some $s \ge a$ and $\sigma \ge 0$, then, by Proposition 3.3, we would have

$$P \in \mathcal{L}(X_{s/2}, X_{-\sigma/2}), \qquad \|P\|_{\mathcal{L}(X_{s/2}, X_{-\sigma/2})} \le C\|d\|_{L^p_U(\mathbb{R}^N)}.$$

Hence we can apply Proposition 10 in [5] with $\alpha = s/2$ and $\beta = -\sigma/2$ provided $0 \le \alpha - \beta < 1$, that is, $s + \sigma < 2$. This result gives a solution of (3.7), $u(t; u_0) = S_P(t)u_0$, $t \ge 0$, satisfying (3.8) for any $\gamma \in E(\alpha) := (\alpha - 1, \alpha]$ and $\gamma' \in R(\beta) := [\beta, \beta + 1)$ with $\gamma' \ge \gamma$. Note that we can always take at least $\gamma, \gamma' \in [\beta, \alpha]$.
Step 2. To determine the set of pairs (s, σ) satisfying (3.6) and $s + \sigma < 2$, we define

$$\tilde{s} = s - a - \frac{N}{q} \quad \text{and} \quad \tilde{\sigma} = \sigma - \frac{N}{q'}, \tag{3.10}$$

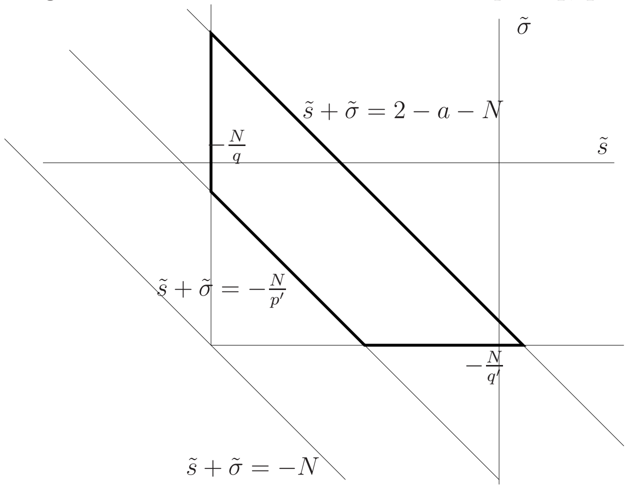
so $\tilde{s} \ge -\frac{N}{q}$, $\tilde{\sigma} \ge -\frac{N}{q'}$ since $s \ge a$, $\sigma \ge 0$. Then (3.6) and $s + \sigma < 2$ read

$$\tilde{s} \ge -\frac{N}{q}, \quad \tilde{\sigma} \ge -\frac{N}{q'}, \qquad -\frac{N}{p'} \le \tilde{s}_- + \tilde{\sigma}_-, \qquad \tilde{s} + \tilde{\sigma} < 2 - a - N. \tag{3.11}$$

Note that this region is nonempty since $-N \le -\frac{N}{p'} < 2 - a - N$ because $p > \frac{N}{2-a}$.

The set of admissible parameters $(\tilde{s}, \tilde{\sigma})$ given by (3.11) depends on the relationship between q, q' and p. Note that (3.11) defines a planar trapezium–shaped polygon, $\tilde{\mathcal{P}}$, whose long base is on the line $\tilde{s} + \tilde{\sigma} = 2 - a - N$ and the short base is on the line $\tilde{s} + \tilde{\sigma} = -\frac{N}{p'}$ in the third quadrant. As for the lateral sides note that the restriction $-\frac{N}{p'} \le \tilde{s}_- + \tilde{\sigma}_-$ adds the condition that $\tilde{s} \ge -\frac{N}{p'}$ in the second quadrant and $\tilde{\sigma} \ge -\frac{N}{p'}$ in the fourth. These have to be combined with $\tilde{s} \ge -\frac{N}{q}$ and $\tilde{\sigma} \ge -\frac{N}{q'}$. Therefore the

Figure 1: Admissible $\tilde{s}$ and $\tilde{\sigma}$ with $p > q, q'$



lateral sides are given by the lines $\tilde{s} = \max\{-\frac{N}{p'}, -\frac{N}{q}\}$ and $\tilde{\sigma} = \max\{-\frac{N}{p'}, -\frac{N}{q'}\}$. One of the possible cases is depicted in Figure 1.

In any case, projecting $\tilde{\mathcal{P}}$ onto the axes gives the following ranges for $\tilde{s}$ and $\tilde{\sigma}$

$$\tilde{s} \in [\max\{-\frac{N}{p'}, -\frac{N}{q}\}, 2 - a - N - \max\{-\frac{N}{p'}, -\frac{N}{q'}\})$$

$$\tilde{\sigma} \in [\max\{-\frac{N}{p'}, -\frac{N}{q'}\}, 2 - a - N - \max\{-\frac{N}{p'}, -\frac{N}{q}\}).$$

Note that, by (3.10), the polygon $\tilde{\mathcal{P}}$ transforms into a similar shaped polygon $\mathcal{P}$ which determines the region of admissible pairs (s, σ). Thus the projection ranges for s and σ are given by

$$s \in J_1 = [a + (\frac{N}{q} - \frac{N}{p'})_+, 2 - (\frac{N}{q'} - \frac{N}{p'})_+) \tag{3.12}$$

$$\sigma \in J_2 = [(\frac{N}{q'} - \frac{N}{p'})_+, 2 - a - (\frac{N}{q} - \frac{N}{p'})_+). \tag{3.13}$$

Step 3. Now we perform a bootstrap argument with the solutions of (3.7).

For any $(s_0, \sigma_0) \in \mathcal{P}$ the line $s + \sigma = s_0 + \sigma_0 := k_0 < 2$ intersects $\mathcal{P}$ along a segment $\mathcal{S}(s_0, \sigma_0)$ which, using (3.12), (3.13), can be parametrized in terms of $s \in J_1(k_0) = [a + (\frac{N}{q} - \frac{N}{p'})_+, k_0 - (\frac{N}{q'} - \frac{N}{p'})_+]$.

Then take $(s, \sigma) \in \mathcal{S}(s_0, \sigma_0)$ with $s \geq s_0$, hence $\sigma \leq \sigma_0$, and such that $s_0 \leq s < 4 - \sigma_0$ which implies that $R(-\frac{\sigma_0}{2}) \cap E(\frac{s}{2}) \neq \emptyset$. Then, using $S_P(t) = S_P(t/2) \circ S_P(t/2)$ and taking $\gamma' \in R(-\frac{\sigma_0}{2}) \cap E(\frac{s}{2}) \neq \emptyset$ with $\gamma' > \gamma$ we get

$$\|S_P(t)u_0\|_{\dot{H}_U^{\gamma'',q}(\mathbb{R}^N)} \leq \frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma''-\gamma'}} \|S_P(t/2)u_0\|_{\dot{H}_U^{\gamma',q}(\mathbb{R}^N)} \leq \tag{3.14}$$

$$\frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma''-\gamma'}}\frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma'-\gamma}}\|u_0\|_{\dot{H}^{\gamma,q}_U(\mathbb{R}^N)} = \frac{Me^{\mu t}}{t^{\gamma''-\gamma}}\|u_0\|_{\dot{H}^{\gamma,q}_U(\mathbb{R}^N)}$$

that is, (3.8) for $\gamma \in E(\frac{s_0}{2}) = (\frac{s_0}{2}-1, \frac{s_0}{2}]$ and $\gamma'' \in R(-\frac{\sigma}{2}) = [-\frac{\sigma}{2}, -\frac{\sigma}{2}+1)$, $\gamma'' > \gamma' > \gamma$, and M depending on γ and γ''.

Note that $s_0 \leq s \leq k_0 - (\frac{N}{q'} - \frac{N}{p'})_+ \leq k_0 < 2 < 4 - k_0 \leq 4 - \sigma_0$ so all conditions above are met. Also, as we take $s \in [s_0, k_0 - (\frac{N}{q'} - \frac{N}{p'})_+]$ and $\sigma = k_0 - s$, we get

$$\gamma'' \in \bigcup_{\{\sigma = k_0 - s,\ s\in[s_0, k_0 - (\frac{N}{q'} - \frac{N}{p'})_+]\}} R(-\frac{\sigma}{2}) = [-\frac{\sigma_0}{2}, 1 - \frac{1}{2}(\frac{N}{q'} - \frac{N}{p'})_+) \tag{3.15}$$

So, as (s_0, σ_0) range in the region $\mathcal{P}$, from (3.15) we get (3.8) for

$$\gamma \in (\frac{\inf J_1}{2} - 1, \frac{\sup J_1}{2}], \qquad \gamma' \in [-\frac{\sup J_2}{2}, 1 - \frac{\inf J_2}{2}), \qquad \gamma' \geq \gamma$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q,a) = (-1 + \frac{a}{2} + \frac{N}{2}(\frac{1}{q} - \frac{1}{p'})_+, 1 - \frac{N}{2}(\frac{1}{p} - \frac{1}{q})_+)$$

which concludes the proof of (3.8).

For the estimates in uniform Lebesgue spaces, (3.9), we use the Sobolev inclusions (2.3). First note that for any $1 < q < \infty$, $I(q,a) \supset (-1 + \frac{a}{2} + \frac{N}{2p}, 1 - \frac{N}{2p})$ which does not depend on q and is not empty because $p > \frac{N}{2-a}$. Let $\tilde{\gamma} := 1 - \frac{N}{2p} > 0$ and take $0 \leq \gamma < \tilde{\gamma}$, then $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N) \hookrightarrow \dot{L}^{\tilde{q}}_U(\mathbb{R}^N)$, for $\tilde{q} \geq q$ such that $-\frac{N}{\tilde{q}} = 2\gamma - \frac{N}{q}$, i.e. $\frac{1}{q} - \frac{1}{\tilde{q}} = \frac{2\gamma}{N}$, and we get

$$\|S_P(t)u_0\|_{\dot{L}^{\tilde{q}}_U(\mathbb{R}^N)} \leq C\|S_P(t)u_0\|_{\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)} \leq \frac{M_\gamma e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{\tilde{q}})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}.$$

In particular we can take $0 \leq \gamma \leq \frac{\tilde{\gamma}}{2}$ and we get the estimate above for all $\tilde{q} \geq q$ such that $\frac{1}{q} - \frac{1}{\tilde{q}} \in [0, \frac{\tilde{\gamma}}{N}]$ and this interval does not depend on q.

Reiterating this argument, starting with $r_0 := q$ and defining the numbers r_i, $i = 1, 2, 3, \ldots$ such that $\frac{1}{r_i} - \frac{1}{r_{i+1}} = \frac{\tilde{\gamma}}{N}$, we obtain the estimate above for any $\tilde{q} \geq q$ such that $\tilde{q} \in [q, r_{i+1}]$. Hence, in a finite number of steps we can reach any $\tilde{q}$ with $q < \tilde{q} \leq \infty$.

The convergence of the semigroups is a direct consequence of [5, Theorem 14] , since Proposition 3.3 gives that if $d_\varepsilon \to d$ in $\dot{L}^p_U(\mathbb{R}^N)$, then $P_\varepsilon \to P$ in $\mathcal{L}(X_{s/2}, X_{-\sigma/2})$ for any pair of admissible $(s, \sigma) \in \mathcal{P}$. The case of Lebesgue spaces follows from this as well.

The order preserving property is obtained by approximation as in Proposition 3.1. From [1], for smooth enough coefficient d and regular initial data, if $u_0 \geq 0$ then $S_P(t)u_0 \geq 0$ for all $t \geq 0$. Now, for $u_0 \in \dot{H}^{\gamma,q}_U(\mathbb{R}^N)$, with $\gamma \in I(q,a)$ and d as in the statement take d_n and $\{u_0^n\}_{n\in\mathbb{N}}$ regular such that $d_n \to d$ in $\dot{L}^p_U(\mathbb{R}^N)$ and $u_0^n \to u_0$ in $\dot{H}^{\gamma,q}_U(\mathbb{R}^N)$. Then $S_{P_n}(t)u_0^n \to S_P(t)u_0$ and therefore $S_P(t)u_0 \geq 0$. Note again that this works because we are working with the "dotted" spaces, where regular functions are dense.

Finally, the analyticity comes from [5, Theorem 12]. ∎

Now, we can combine several perturbations simultaneously.

Remark 3.5 *For the problem*

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0) = u_0 \end{cases}$$

with $b_j \in \dot{L}^{p_j}_U(\mathbb{R}^N)$ with $p_j > N$, since the uniform Lebesgue spaces are nested we have that $b_j \in \dot{L}^p_U(\mathbb{R}^N)$ with $p = \min_{j=1,\dots,N}\{p_j\} > N$ and then $P = \sum_{j=1}^N b_j(x)\partial_j$ satisfies Proposition 3.3 with such p and $a = 1$. Thus Theorem 3.4 remains valid for the problem above.

When we combine zeroth and first order terms, we get the following result which proves Theorem 1.1.

Theorem 3.6 *Assume for $j = 1, \dots, N$,*

$$\|b_j\|_{\dot{L}^{p_j}_U(\mathbb{R}^N)} \leq R_j \quad \text{and} \quad \|c\|_{\dot{L}^{p_0}_U(\mathbb{R}^N)} \leq R_0$$

where $p_j > N$ and $p_0 > \frac{N}{2}$. Define $a_0 = 0$, $a_j = 1$ and for $j = 1, \dots, N$ and, $\tilde{p} = \min\{p_j,\ j = 1, \dots, N\} > N$. If $q' < \tilde{p}$ and $q > p_0$, we will also assume $p_0 > \frac{Nq}{N+q}$.

Then for any $1 < q < \infty$ there exists non-empty interval $I(q) \subset (-\frac{1}{2}, 1)$ containing $(-1 + \max_j\{\frac{a_j}{2} + \frac{N}{2p_j}\}, 1 - \max_j\{\frac{N}{2p_j}\})$, such that for any $\gamma \in I(q)$, we have a strongly continuous, order preserving, analytic semigroup $S(t)$, in the space $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$, for the problem

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) \qquad x \in \mathbb{R}^N \end{cases} \tag{3.16}$$

with $u(t; u_0) = S(t)u_0$, $t \geq 0$.

Moreover the semigroup has the smoothing estimate

$$\|S(t)u_0\|_{\dot{H}^{2\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{2\gamma}_U(\mathbb{R}^N) \tag{3.17}$$

for every $\gamma, \gamma' \in I(q)$ with $\gamma' \geq \gamma$, and

$$\|S(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N) \tag{3.18}$$

for $1 < q \leq r \leq \infty$ with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on M, θ_0, R_j and R_0.

Furthermore,

$$I(q) = (-1 + \max_{j=0,\dots,N}\{\frac{a_j}{2} + \frac{N}{2}(\frac{1}{p_j} - \frac{1}{q})_+\}, 1 - \frac{N}{2}(\frac{1}{\min_{j=0,\dots,N}\{p_j\}} - \frac{1}{q})_+). \tag{3.19}$$

Finally, if, as $\varepsilon \to 0$

$$b_j^\varepsilon \to b_j \quad \text{in } \dot{L}^{p_j}_U(\mathbb{R}^N), \quad p_j > N, \quad j = 1, \dots, N,$$

$$c^\varepsilon \to c \quad in\ \dot{L}^{p_0}_U(\mathbb{R}^N), \quad p_0 > N/2$$

then for every $T>0$ there exists $C(\varepsilon)\to 0$ as $\varepsilon \to 0$, such that

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(\dot{H}^{2\gamma,q}_U(\mathbb{R}^N), \dot{H}^{2\gamma',q}_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0<t\leq T$$

for all $\gamma, \gamma' \in I(q)$, $\gamma' \geq \gamma$ and for all $1<q\leq r\leq \infty$,

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^r_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0<t\leq T.$$

Proof. Consider the lower order terms as perturbations $P_j u := b_j \partial_j u$, $P_0 u := cu$. As in the proof of Theorem 3.4, for each perturbation P_j there exists a non empty trapezoidal polygon $\mathcal{P}_j$ of admissible pairs of spaces (s,σ) described in terms of $\tilde{s} = s - a_j - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - \frac{N}{q'}$, see (3.11).

According to Lemma 13, iii) in [5], we can consider $P := \sum_{j=0}^N P_j$, that is, all perturbations acting at the same time, if there exists a common region $\mathcal{P}$ of admissible pairs (s,σ), that is if $\mathcal{P} := \cap_{j=0}^N \mathcal{P}_j \neq \emptyset$.

Recall from the proof of Theorem 3.4 that the polygon $\mathcal{P}_j$ of the perturbation P_j is given by a planar trapezium whose long base is on the line $s+\sigma = 2$ and the short base is on the line $s+\sigma = a_j + \frac{N}{p_j}$ in the third quadrant. Also, the lateral sides they are given by the lines $s = a_i + (\frac{N}{q} - \frac{N}{p_j'})_+$ and $\sigma = (\frac{N}{q'} - \frac{N}{p_j'})_+$. Thus the projection of $\mathcal{P}_j$ on the axes give the intervals

$$s \in J_1^j = [s^j_{min}, 2-\sigma^j_{min}) \quad and \quad \sigma \in J_2^j = [\sigma^j_{min}, 2 - s^j_{min})$$

see (3.12) and (3.13). Therefore, the set $\mathcal{P}$ is non empty if and only if

$$\max_j\{\inf J_1^j\} < \min_j\{\sup J_1^j\} \quad i.e. \quad \max_j\{s^j_{min}\} < \min_j\{2-\sigma^j_{min}\}$$

and

$$\max_j\{\inf J_2^j\} < \min_j\{\sup J_2^j\} \quad i.e. \quad \max_j\{\sigma^j_{min}\} < \min_j\{2-s^j_{min}\}$$

which are equivalent to $\max_j\{s^j_{min}\} + \max_j\{\sigma^j_{min}\} < 2$, that is,

$$\max_{j=0,\dots,N}\{a_j + (\frac{N}{p_j} - \frac{N}{q'})_+\} + \max_{j=0,\dots,N}\{(\frac{N}{p_j} - \frac{N}{q})_+\} < 2. \tag{3.20}$$

We prove below that this condition is always satisfied; see Lemma 3.9.

Assuming this for a while, the projection of $\mathcal{P} = \bigcap_{j=0}^N \mathcal{P}_j$ on the axes gives the intervals

$$s \in J_1 = [\max_j(\inf J_1^j), \min_j(\sup J_1^j)) = [\max_j\{a_j + (\frac{N}{p_j} - \frac{N}{q'})_+\}, 2 - \max_j\{(\frac{N}{p_j} - \frac{N}{q})_+\})$$

$$\sigma \in J_2 = [\max_j(\inf J_2^j), \min_j(\sup J_2^j)) = [\max_j\{(\frac{N}{p_j} - \frac{N}{q})_+\}, 2 - \max_j\{a_j + (\frac{N}{p_j} - \frac{N}{q})_+\}).$$

For each pair of admissible pairs $(s,\sigma)\in\mathcal{P}$, by [5, Proposition 10] (see the proof of Theorem 3.4) with $\alpha=\frac{s}{2}$ and $\beta=-\frac{\sigma}{2}$, we get a solution of (3.16) satisfying (3.17) for

$$\gamma\in E(\alpha)=(\alpha-1,\alpha], \qquad \gamma'\in R(\beta)=[\beta,\beta+1), \qquad \gamma'\geq\gamma.$$

Hence as (s,σ) range in the region $\mathcal{P}$ a repeated bootstrap argument as in (3.14) gives that the smoothing estimates hold for $\gamma\in\bigcup_{(s,\sigma)\in\mathcal{P}}E(s/2)$ and $\gamma'\in\bigcup_{(s,\sigma)\in\mathcal{P}}R(-\sigma/2)$, $\gamma'\geq\gamma$. This leads to

$$\gamma\in(\frac{\inf J_1}{2}-1,\frac{\sup J_1}{2}], \qquad \gamma'\in[-\frac{\sup J_2}{2},1-\frac{\inf J_2}{2}), \quad \gamma'\geq\gamma$$

which, after a simple calculation, reads

$$\gamma,\gamma'\in I(q)=(-1+\max_j\{\frac{a_j}{2}+\frac{N}{2}(\frac{1}{p_j}-\frac{1}{q'})_+\},1-\max_j\{\frac{N}{2}(\frac{1}{p_j}-\frac{1}{q})_+\}),$$

which gives (3.19). Note that this interval is contained in the interval $(-\frac{1}{2},1)$ and contains $(-1+\max_j\{\frac{a_j}{2}+\frac{N}{2p_j}\},1-\max_j\{\frac{N}{2p_j}\})$, which is independent of q and non empty because $p_j>\frac{N}{2-a_j}$. To see this note that the latter condition gives $\frac{a_j}{2}+\frac{N}{2p_j}<1$ and $\frac{N}{2p_j}<1-\frac{a_j}{2}<1$.

The estimates in uniform Lebesgue spaces (3.18) are obtained using Sobolev embeddings as in Theorem 3.4.

The order preserving property is obtained by approximation with smooth coefficients and initial data as in Theorem 3.4. Finally, the analyticity comes from [5, Theorem 12]. ■

Remark 3.7
i) Note that the interval in (3.19) is in fact the intersection of the intervals of each separate perturbation as obtained in Theorem 3.4.
ii) The additional assumption in Theorem 3.6 that "if $q'<\tilde{p}$ and $q>p_0$, we will also assume $p_0>\frac{Nq}{N+q}$" applies only when q is large relative to the exponents p_j, $j=0,\ldots,N$.

Also, for $1<q\leq N$ this imposes no additional restriction since in this range $\frac{Nq}{N+q}\leq\frac{N}{2}$. Furthermore, since $\frac{Nq}{N+q}<N$ for all q, if $p_0\geq N$ no additional assumption is imposed either.

Remark 3.8 *If we assume that $p_j\geq q$ for $j=0,\ldots,N$ as in Theorem 5.3 in [3], then Theorem 3.6 applies and we get in (3.19) an interval*

$$I(q)=(-1+\max_{j=0,\ldots,N}\{\frac{a_j}{2}+\frac{N}{2}(\frac{1}{p_j}-\frac{1}{q})_+\},1).$$

Since this interval contains 0, then Theorem 3.6 recovers Theorem 5.3 in [3] and improves it since in (3.17) we can even take γ slightly negative.

This case includes the case in which b_j and c_0 are bounded functions.

Now we prove our claim about (3.20).

Lemma 3.9 *With the assumptions in Theorem 3.6, condition (3.20) is satisfied.*

Proof. Observe that denoting $\tilde{p} = \min\{p_j,\ j = 1, \dots, N\} > N$ and $p = \min\{p_j,\ j = 0, \dots, N\} = \min\{p_0, \tilde{p}\} > \frac{N}{2}$, then (3.20) can be written as

$$\max\{(\frac{N}{p_0} - \frac{N}{q'})_+, 1 + (\frac{N}{\tilde{p}} - \frac{N}{q'})_+\} + \max\{(\frac{N}{p_0} - \frac{N}{q})_+, (\frac{N}{\tilde{p}} - \frac{N}{q})_+\} < 2.$$

To prove the lemma we prove that all for possible sums of the terms inside the "max" above are less than 2.

1. Let $M = (\frac{N}{p_0} - \frac{N}{q'})_+ + (\frac{N}{p_0} - \frac{N}{q})_+$
   (a) If $q, q' < p_0$ then $M = 0 < 2$.
   (b) If $q < p_0 < q'$ then $M = \frac{N}{p_0} - \frac{N}{q'} < \frac{N}{p_0} < 2$.
   (c) If $q' < p_0 < q$ then $M = \frac{N}{p_0} - \frac{N}{q} < \frac{N}{p_0} < 2$.
   (d) If $p_0 < q, q'$ then $M = \frac{2N}{p_0} - N = \frac{N}{p_0} - \frac{N}{p_0'} < \frac{N}{p_0} < 2$.
2. Let $M = (\frac{N}{p_0} - \frac{N}{q'})_+ + (\frac{N}{\tilde{p}} - \frac{N}{q})_+$
   (a) If $q' < p_0$ and $q < \tilde{p}$ then $M = 0$.
   (b) If $p_0 < q'$ and $q < \tilde{p}$ then $M = \frac{N}{p_0} - \frac{N}{q'} < \frac{N}{p_0} < 2$.
   (c) If $q' < p_0$ and $q > \tilde{p}$ then $M = \frac{N}{\tilde{p}} - \frac{N}{q} < \frac{N}{\tilde{p}} < 1$.
   (d) If $p_0 < q'$ and $q > \tilde{p}$ then $M = \frac{N}{p_0} + \frac{N}{\tilde{p}} - N = \frac{N}{\tilde{p}} - \frac{N}{p_0'} < \frac{N}{\tilde{p}} < 1$,
3. Let $M = 1 + (\frac{N}{\tilde{p}} - \frac{N}{q'})_+ + (\frac{N}{p_0} - \frac{N}{q})_+$
   (a) If $q' < \tilde{p}$ and $q < p_0$ then $M = 1$.
   (b) If $\tilde{p} < q'$ and $q < p_0$ then $M = 1 + \frac{N}{\tilde{p}} - \frac{N}{q'} < 1 + \frac{N}{\tilde{p}} < 2$.
   (c) If $q' < \tilde{p}$ and $q > p_0$ then $M = 1 + \frac{N}{p_0} - \frac{N}{q} < 2$ because $p_0 > \frac{Nq}{N+q}$ by assumption.
   (d) If $\tilde{p} < q'$ and $q > p_0$ then $M = 1 + \frac{N}{\tilde{p}} + \frac{N}{p_0} - N = 1 + \frac{N}{\tilde{p}} - \frac{N}{p_0'} < 1 + \frac{N}{\tilde{p}} < 2$.
4. Let $M = 1 + (\frac{N}{\tilde{p}} - \frac{N}{q'})_+ + (\frac{N}{\tilde{p}} - \frac{N}{q})_+$
   (a) If $q', q < \tilde{p}$ then $M = 1$.
   (b) If $q < \tilde{p} < q'$ then $M = 1 + \frac{N}{\tilde{p}} - \frac{N}{q'} < 1 + \frac{N}{\tilde{p}} < 2$.
   (c) If $q' < \tilde{p} < q$ then $M = 1 + \frac{N}{\tilde{p}} - \frac{N}{q} < 1 + \frac{N}{\tilde{p}} < 2$.
   (d) If $\tilde{p} < q, q'$ then $M = 1 + \frac{2N}{\tilde{p}} - N = 1 + \frac{N}{\tilde{p}} - \frac{N}{\tilde{p}'} < 1 + \frac{N}{\tilde{p}} < 2$.

■

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