

**PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2010-16**

**Sensitivity Analysis of a Default Time Model
for Credit Risk Portfolio Management**

R. Abella Muñoz, I. Armero Huertas,
B. Ivorra and A. M. Ramos del Olmo

Noviembre-2010

<http://www.mat.ucm.es/deptos/ma>
e-mail:matemática_aplicada@mat.ucm.es

Sensitivity Analysis of a Default Time Model for Credit Risk Portfolio Management

Rebeca Abella Muñoz, Ismael Armero Huertas
Benjamin Ivorra, Ángel Manuel Ramos del Olmo

Departamento de Matemática Aplicada,
Facultad de C.C. Matemáticas,
Universidad Complutense de Madrid,
3 Plaza de Ciencias, 28040, Madrid, Spain

November 25, 2010

1 Risk Modeling Introduction

Credit risk is described as the risk of trading partners, named counterparties, not fulfilling their obligations on the due date or at any time thereafter resulting into losses for investors [13]. This situation can be generated by many factors such as credits events (frauds, bankruptcy, etc.) or variations of counterparty's rating.

The principal objective of the credit risk management is to provide risk models and evaluation tools in order not only to evaluate risk on financial products, but also to intend to control it. The main difficulty is the complexity of the loss distribution. Credit losses are characterized by large probabilities of small earnings, combined with a small chance of losing a big percentage of the amount of the investments. As a result, the loss distribution is heavily asymmetric and functions related with risk measure are highly non-linear.

A **portfolio** is defined as a collection of investments all owned by the same individual or organization. These investments can include stocks (investments in individual businesses), bonds (investments in debt that are

designed to earn interest) and mutual funds (pools of money from many investors that are invested by professionals or according to indices).

Our principal aim is to provide a method in order to evaluate and avoid losses in the portfolio. We analyze the portfolio's tendency depending on the changes of the variables involved in the models such as risk or nominal. First of all, we study the behaviour of the portfolio's losses in the general case. Afterwards, we modify each parameter separately in order to reach some conclusions about each variable's sensitivity. And finally, all parameters are changed at the same time.

A glossary will be added to the end of the document in order to help the reader to better understand the meaning of certain terms.

2 Collateralized Loan Obligation

Collateralized Loan Obligations (CLO)

CLO are security interests in pools of assets that usually comprise loan. The objective, for financial institutions, is to buy securitization in order to protect themselves from eventual defaults of counterparties included in the CLO [12].

Investors support the credit risk of the collateral but in counterpart receive, until CLO maturity date, a periodic remuneration proportional to the risk.

We present here the general structure of the portfolio to be optimized. In fact, we consider a portfolio of portfolios, which are called inner portfolio.

Numerous tranches of securities are originated by CLO's, offering investors multiple credit risk characteristics. Tranches are divided in categories according to their degree of risk:

- Senior: Low credit risk. It is a security that only covers high loss events. Consequently, the spread paid by investors is the lowest.
- Mezzanine: Medium credit risk. Intermediate loss events are covered.
- Junior: High credit risk. In this case the spread is the highest as the security covers first losses. Usually, due to this high value, financial institutions do not buy securitization on this tranche.

CLO's structure example

We illustrate the CLO's structure with this example, we consider a maturity of one year and compound by 4 facilities:

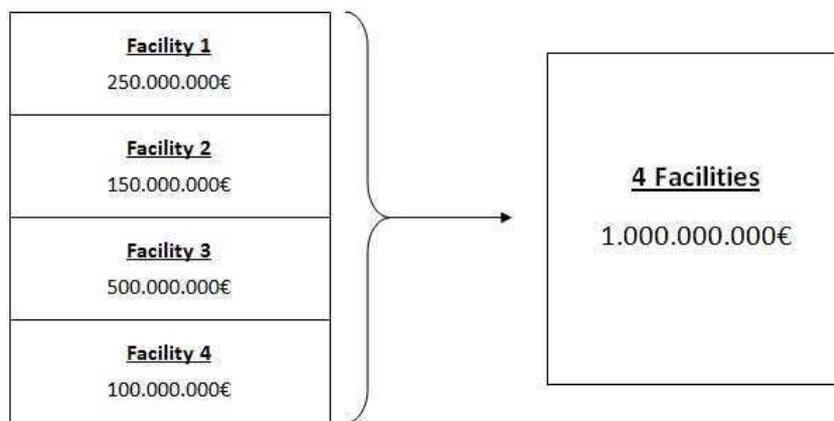


Figure 1: CLO's example

We have a nominal amount of 1.000.000.000€

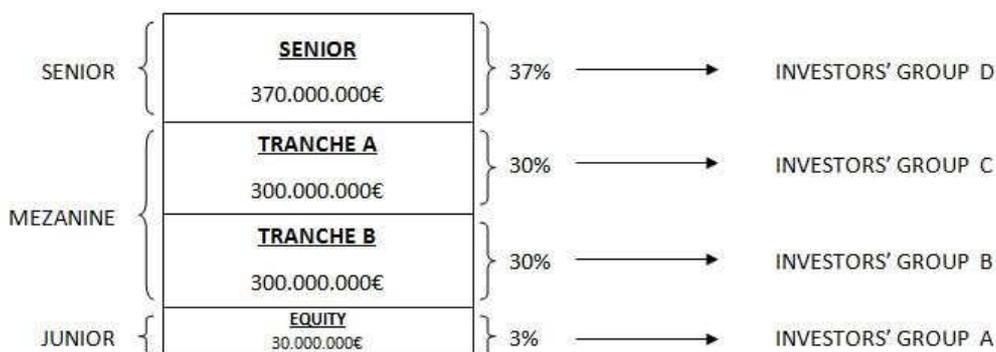


Figure 2: CLO's example

The monthly remuneration that that investors receive to cover each tranche is:

- Investors' group A, they cover the equity

They receive 600 b.p. → 1.800.000 €/month.

- Investors' group B, they cover the Tranche B

They receive 50 b.p. \longrightarrow 1.500.000 €/month.

- Investors' group C, they cover the Tranche C

They receive 20 b.p. \longrightarrow 600.000 €/month.

- Investors' group D, they cover the Senior

They receive 5 b.p. \longrightarrow 185.000 €/month.

The amount paid to the investors is 4.085.000 €/month, in one year will be 49.020.000 €. In case of defaults, there are four different scenarios:

1. $Loss \leq 3\%$. Only investors who are in group A will pay to financial institution, they will pay the CLO's loss:

GROUP A \longrightarrow (CLO's loss).

2. $3\% < Loss \leq 33\%$. Investors who are in group A will pay 3% of CLO's nominal and investors who are in group B will pay the rest (30% of CLO's nominal) $- [CLO's loss - (3\% \text{ of } CLO's \text{ nominal})]$

GROUP A \longrightarrow 30.000.000€

GROUP B \longrightarrow $[CLO's loss - (30.000.000€)]$.

3. $33\% < Loss \leq 63\%$. Investors who are in group A will pay 3% of CLO's nominal, investors who are in group B will pay 30% of CLO's nominal and the rest will be paid by investors who are in group C

GROUP A \longrightarrow 30.000.000€

GROUP B \longrightarrow 300.000.000€

GROUP C \longrightarrow $[CLO's loss - (330.000.000€)]$.

4. $63\% < Loss$. Investors who are in group A will pay 3% of CLO's nominal, investors who are in group B and group C will pay 30% of CLO's nominal and the rest will be paid by investors who are in group D

GROUP A	→	30.000.000€
GROUP B	→	300.000.000€
GROUP C	→	300.000.000€
GROUP C	→	$[CLO's\ loss - (630.000.000€)]$.

3 Mathematical modeling

3.1 Historical background

There have been a long amount of studies about Collateral Loan Obligation which have been developed through history. This is why there are various mathematical background about this subject. Consequently, we have found a lot of literature related with the algorithm that we use and briefly explain below.

3.1.1 Structural Models

There are two principal types of models that try to describe default processes in credit risk literature: structural and reduced form models.

Structural models make use of the information provided by the evolution of the firm's structural variables, such as default and firm value, to find out the time of default. In this instance, it is an endogenous process. On the other hand, reduced models define the time of default as the first jump of an exogenously given jump process in which the parameters are given by the market data. Consequently, reduced models rely on external parameters while structural models are characterized, not only because of its internal process, but also because of providing a relation between the credit quality and the firm's economic and financial conditions. Another difference between the two approaches mentions the treatment of the recovery rate: in reduced models the recovery is not determined by the value of the firm's assets and liabilities, but specified by exogenous factors. Moreover, structural models have considered interest rates.

Merton's Model [15], which can be defined as the beginning of the structural literature on credit risk, applies a theory developed by Black and Scholes [6], to the modeling of a firm's debt. In this method, the firm defaults if its assets are below the debt at the time of maturity.

Merton characterized the firm's capital structure by equity and a zero-coupon bond with maturity T and value D , so the debt value will be just the difference between the asset v and the equity value.

In a second approach, written by Black and Cox (1976) [5], defaults take place as soon as the firm's asset value is below a certain threshold. This paper is considered the first of the First Passage Models (FPM). First Passage Models describe the default as the first time the firm's debt value falls below a lower barrier and the firm is liquidated immediately after the event. When the default event does not directly mean liquidation, but it is regarded as the beginning of a process, the model is considered to be part of the Liquidation Process Models (LPM). Finally, State Depend Models (SDM) assume that some of the parameters are depend on the business cycle or the firm's external rating.

Merton's Model

As it was mentioned before, the capital structure was formed by equity and a zero-coupon bond, also known as an accrual bond, which is a debt security that does not pay interest, but is traded at a deep discount rendering profit at maturity when the bond is redeemed for its full face value.

We define V_T as the firm's value asset and D as zero coupon bond value. Therefore, at time of maturity T only two options are available:

- $V_T > D \Rightarrow$ The firm does not default and shareholders receive $V_T - D$.
- $V_T < D \Rightarrow$ The firm default and bonholders take control of the firm.

Merton assumes the inexistence of transaction costs, bankruptcy costs, taxes or problems with indivisibilities of assets. He takes the interests rates as a constant r and assumes that the value of the firm is invariant under changes in its capital structure.

The firm's asset value follow a diffusion process:

$$dV_t = rV_t dt + \sigma_V V_t dW_t,$$

where σ_V is the relative asset volatility and W_t is a Brownian motion [4].

The payoffs to equityholders are $E_T = \max\{V_T - D, 0\}$, and to bondholders are $z(T, T) = V_T - E_T$.

When we apply the Black-Scholes pricing formula, the value of equity at time t ($0 \leq t \leq T$) is:

$$E_t(V_t, \sigma_V, T - t) = e^{-r(T-t)} [e^{r(T-t)} V_t \Phi(d_1) - D \Phi(d_2)],$$

where $\Phi(\cdot)$ is the distribution function of a standard normal random variable and d_1 and d_2 are given by:

$$d_1 = \frac{\ln\left(\frac{e^{r(T-t)} V_t}{D}\right) + \frac{1}{2} \sigma_V^2 (T - t)}{\sigma_V \sqrt{T - t}}$$

$$d_2 = d_1 - \sigma_V \sqrt{T - t}.$$

Finally, the probability of default at time T is given by $P[V_T < D] = \Phi(-d_2)$, and the value of the debt at time t obviously is $z(t, T) = V_t - E_t$.

To apply Merton's model we have to estimate:

- V_t : the firm's asset value.
- σ_V : its volatility.
- $z(t, T)$: the zero-coupon bond.

Apart from the restriction of the default time, Merton's Model has a number of other disadvantages. One of the problems refers to the capital structure as it is much more complicated than a simple zero-coupon bond. The rest of assumptions Merton adopts are also nonrealistic, such as the inexistence of transaction or bankruptcy costs, taxes, invariant value of the firm, etc. But the major handicap of the model is that Merton assumes a constant and flat term of interest rate.

First Passage Models

First Passage Models were introduced by Black and Cox [5] extending Merton's model to the case when the firm may default at any time, not only at the maturity date of the debt.

The firm's asset value follow the same diffusion process that Merton's model $dV_t = rV_t dt + \sigma_V V_t dW_t$, but Black and Cox considered a constant default threshold $K > 0$, and the time of default τ is given by:

$$\tau = \inf \{s \geq t | V_s \leq K\}.$$

Therefore, while we are at time $t \geq 0$, default has not been triggered and $V_t > K$.

If we use the reflection principle of the Brownian motion W_t we can infer the default probability from time t to T :

$$P[\tau \leq T | \tau > t] = \Phi(h_1) + \exp\left\{2\left(r - \frac{\sigma_V^2}{2}\right) \ln\left(\frac{K}{V_t}\right) \frac{1}{\sigma_V^2}\right\} \Phi(h_2),$$

where

$$h_1 = \frac{\ln\left(\frac{K}{e^{r(T-t)} V_t}\right) + \frac{\sigma_V^2}{2}(T-t)}{\sigma_V \sqrt{T-t}},$$

$$h_2 = h_1 - \sigma_V \sqrt{T - t}.$$

In order to solve the major criticism Merton's model has received, FPM introduce a default threshold so that the firm may default at any time, not only at the maturity date of the debt. If we assume a constant threshold K , the time of default is given by the first time before maturity that the asset value is below the constant K . However, Black and Cox defined a time dependent default threshold.

The default threshold can be understood in different ways. It can be interpreted as a safety guarantee of the firm's debt so bondholders take control of the company once its asset value reaches this level. In this case, the default threshold would be exogenously established. But it can also be considered endogenously if the stockholders set the value in order to maximize the equity. Another possibility is the negotiation between stockholders and bondholders to determine the default threshold.

These kinds of models consider some extensions in order to introduce more realism: strategic default, debt subordination, stochastic interest rates, etc. but, on the other hand, they increment its analytical complexity.

It also has many drawbacks. The main one is the analytical complexity that makes difficult to obtain closed form expressions for the value of the firm's equity and debt. Another drawback is the predictability of defaults, because default does not come as a surprise, it makes it a predictable event and it allows investors the perfect knowledge of the firm's asset value and default threshold.

Default Correlation

After the single firm case has been mentioned, we present structural models for default correlation between firms. There are several ways to introduce default dependences correlating the firms' asset processes. If the dependence is described as the firm's credit quality on common macroeconomic factors, it is called **cyclical default correlation**. However, it does not account for all the risk dependence between firms. To introduce default dependences between firms in structural models we have to correlate the firms' asset processes. We suppose that we have $i = 1, \dots, I$ different firms with asset value processes given by

$$dV_{i,t} = rV_{i,t}dt + \sigma_{V_i}V_{i,t}dW_{i,t},$$

where $W_{1,t}, \dots, W_{i,t}$ are correlated Brownian motions. Since this model imply predictable defaults, we introduce correlated jumps components.

Giesecke and Goldberg [11] studied structural models for default correlation and took into account credit risk contagion effects in what they termed **contagion default correlation** where the default of one firm can cause the default of linked firms. Therefore, they illustrate the relation between firms in terms of, for example, financial or commercial characteristics so that the firms' default thresholds depend on each other.

Finally, we describe factor models which divide the firm's asset values in groups of common factors which introduce the default correlation in the model and the firm's particular factor.

3.1.2 Factor Models

A factor model relates the systematic or non-diversifiable components of the economy that drive changes in the asset value. The most generic form of this model responds to the formula

$$V_{i,t} = \sum_{j=1}^J w_{i,j} Z_{j,t} + \epsilon_{i,t},$$

where $V_{i,t}$ is the return value on asset i , $w_{i,j}$ is the change in return on asset i per unit change in factor j , $Z_{j,t}$ is the value of factor j and $\epsilon_{i,t}$ is the portion of the return on asset i , not related to the factors. In the formula, $Z_{j,t}$ and $\epsilon_{i,t}$ are unknown, while $w_{i,j}$ is a deterministic parameter.

As we have mentioned before, default correlation is used to measure the default relationship between two firms, either positive or negative. If there is no sign of such relationship, it means that the default are independent. In this case, the default probability of both firms is the product of the individual probabilities of default.

In literature, negative default of correlation has often not been considered because of its lack of occurrence. Besides, we have considered the case in which if two borrowers are correlated, it does not mean directly that the probability of defaulting at the same time is higher. It is also possible that the default of the firm can benefit other firms.

Thus, if we have to evaluate the degree of relationship between the variables of the factor models, that is, if we have to define the correlation, it must be follow that:

- $corr(\epsilon_i, Z_j) = 0$ which means that the value of the factor j $Z_{j,t}$ and the portion of the return on asset i E_i not related with the factors are independent.

- $corr(\epsilon_i, \epsilon_j) = 0$ due to the fact that the portions of the return on different assets have no relation at all.

One example of a Factor Model is the KMV's Model [16], is a methodology proposed by Kealhofer, McQuown and Vasicek. The KMV's principle refers to the correlations of default between clients that are presented using a type of structural default processes called factor models.

KMV's Model

The determination of default correlation has been one of the major problems in portfolio management of default risk. Although it exists a lot of historical information about the relationship between firms, and it is possible to estimate an average default correlation, the estimates obtained are highly inaccurate.

The derivatives approach enables us to determinate the default correlation between two firms knowing their asset correlation and their individual asset probabilities. Extending this model to estimate the value correlation between each pair of firms in the portfolio, and obtain the correlation matrix Σ we require KMV's Model. Extending this model to estimate the value correlation between each pair of firms in the portfolio, and not only the correlation between two variables, we require the correlation matrix. This matrix represents the estimations of the relationships among all the clients in the portfolio, i.e. between all possible pairs of variables. In this instance, we will obtain the correlation matrix Σ from KMV's Portfolio Manager.

The KMV model proposes using a factor model, but instead of having unknown factors, they are taken as observed. KMV's model assumes that the firm's return volatility can be explained by two effects: a systematic effect, defined by the composite factor and particular effects, characterized by the firm.

$$\begin{bmatrix} \text{Firm} \\ \text{Return} \end{bmatrix} = \begin{bmatrix} \text{Composite} \\ \text{Factor} \\ \text{Return} \end{bmatrix} + \begin{bmatrix} \text{Firm} \\ \text{Specific} \\ \text{Effects} \end{bmatrix}$$

We can decompose the composite factor return as:

$$\begin{bmatrix} \text{Composite} \\ \text{Factor} \\ \text{Return} \end{bmatrix} = \begin{bmatrix} \text{Country} \\ \text{Factor} \\ \text{Returns} \end{bmatrix} + \begin{bmatrix} \text{Industry} \\ \text{Factor} \\ \text{Returns} \end{bmatrix}$$

In order to identify the firm's return assets sensibility of the global, regional and sectorial systematic movements, the industry and country rates are discomposed in global, regional and sectorial factors.

Each client is identified by 14 random variables $((G^k)_{1 \leq k \leq 14})$: 2 global rates, 5 regionals and 7 sectorials, by their activity sectors (61 sectors) and their geographical areas (45 areas) using random variables C_i and I_i respectively. Those factors G_i^k , C_i and I_i are supposed to be independent and normally distributed.

Finally, the correlation between clients in this method is given by:

$$\text{corr}(i, j) = R_i R_j \left[\sum_{k=1}^{14} \alpha_k^i \alpha_k^j + \beta^{C_i} \beta^{C_j} + \beta^{I_i} \beta^{I_j} \right] + \sqrt{1 - R_i^2} \sqrt{1 - R_j^2} \delta_i^j,$$

where $\delta_i^j = 1$ if $i = j$, 0 otherwise. α_k^i, β^{C_i} and β^{I_i} are real coefficients that represent the dependence of each firm to the factors G^k , C_i and I_i respectively and such that $\sum_{k=1}^{14} (\alpha_k^i)^2 + (\beta^{C_i})^2 + (\beta^{I_i})^2 = 1$.

3.1.3 Parameters Estimation

There are several ways of calibrating V_t and σ_V

- Making use of Itô's Lemma to obtain a system of two equations in which we only need to know the variables V_t and σ_V [7]. Assume that the firm's equity value follows a geometric Brownian motion under \mathbb{P} , with volatility σ_E

$$\begin{cases} \sigma_V = \frac{E_t}{V_t} \sigma_E \Phi(d_1) \\ E_t(V_t, \sigma_V, T - t) = \hat{E}_t, \end{cases}$$

where \hat{E}_t is the observed market price.

- Duan [2] proposes another method based on maximum likelihood estimation using equity prices and the one-to-one relationship between equity and asset levels given by:

$$E_t(V_t, \sigma_V, T - t) = e^{-r(T-t)} [e^{r(T-t)} V_t \Phi(d_1) - D \Phi(d_2)].$$

- Jones [14] propose a different way of estimating V_t and σ_V that consists on estimating the asset value as the sum of the equity market value, the market value of traded debt and the estimated value of non-traded debt. Making a time series for V_t we can estimate its volatility σ_V .

3.2 Considered Model

In this section, we present the basis of the model used to evaluate CLO's Loss density function. The main idea is to obtain a set of possible scenarios (Monte Carlo algorithm). Therefore, we use a default time model to simulate those times for all the firms in our portfolio (Copula function) and taking into account possible correlations between each pair of firms (KMV's Model).

3.2.1 Copula functions

We have used the probabilistic concept of copula in order to obtain the firm's default time. The main idea is to define the marginal distribution of survival time for each credit risk. In order to determine the joint distribution of survival times with given marginal distributions and a correlation structure, we used the Copula function as a simple and convenient approach.

A copula function links univariate marginals with their multivariate distribution. For n random variables, u_1, u_2, \dots, u_n , the default correlation parameter ρ , the joint distribution function C , also called Copula function, will be define as

$$C(u_1, u_2, \dots, u_n, \rho) = P[U_1 \leq u_1, U_2 \leq u_2, \dots, U_n \leq u_n]$$

If we apply it to link marginal distributions with a joint distribution, for a given univariate marginal distribution functions $F_1(x_1), F_2(x_2), \dots, F_n(x_n)$, the function

$$C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)) = F(x_1, x_2, \dots, x_n)$$

results in a multivariate distribution function with univariate marginal distributions:

$$\begin{aligned} C(F_1(x_1), \dots, F_n(x_n), \rho) &= P[U_1 \leq F_1(x_1), U_2 \leq F_2(x_2), \dots, U_n \leq F_n(x_n)] \\ &= P[F_1^{-1}(U_1) \leq x_1, \dots, F_n^{-1}(U_n) \leq x_n] \\ &= P[X_1 \leq x_1, \dots, X_n \leq x_n] \\ &= F(x_1, x_2, \dots, x_n). \end{aligned}$$

Thus, the marginal distribution of X_i is

$$\begin{aligned} C(F_1(+\infty), \dots, F_n(+\infty), \rho) &= P[X_1 \leq +\infty, \dots, X_n \leq +\infty] \\ &= P[X_i \leq (x_i)] \\ &= F_i(x_i). \end{aligned}$$

Suppose a portfolio with n clients. Let us assume that for each client i we have constructed the distribution function $F_i(t)$ of survival time T_i . Using a copula function C we also obtain the joint distribution of the survival times as follows:

$$F(t_1, t_2, \dots, t_n) = (C(F_1(t_1), F_2(t_2), \dots, F_n(t_n)), \rho).$$

But using normal copula function, we have

$$F(t_1, t_2, \dots, t_n) = \Phi_n(\Phi^{-1}(F_1(t_1)), \Phi^{-1}(F_2(t_2)), \dots, \Phi^{-1}(F_n(t_n))),$$

where Φ_n is the n dimensional normal cumulative distribution function with correlation matrix Σ . Finally, introducing another series of random variables Y_1, Y_2, \dots, Y_n we will obtain the correlated survival times:

$$Y_1 = \Phi^{-1}(F_1(T_1)), Y_2 = \Phi^{-1}(F_2(T_2)), \dots, Y_n = \Phi^{-1}(F_n(T_n)).$$

With each simulation we create the survival times for all the credits in the portfolio. Consequently, we are able to value any credit derivative structure. Our algorithm is based on the Monte Carlo simulation approach and the normal copula function to define the survival time distribution. For each simulation we create a possible scenario of default times t_1, t_2, \dots, t_n , from which we have the first-to-default time as the minimum of the default times.

Example of a Copula model

We introduce an example of the model explained above in which we are able to determine the joint distribution of survival times. First of all, we simulate a correlation structure with the Sigma-Matrix: $\Sigma = \Xi^t \Xi$, where Σ represent the correlation between each client in the considered portfolio and Ξ its Cholesky's decomposition. Afterwards, we create the marginal distributions $V = \Xi G = (v_1, \dots, v_n)$ to obtain the joint distribution of the survival times $F(t_1, t_2, \dots, t_n) = C(F_1(t_1), F_2(t_2), \dots, F_n(t_n))$, as follows: $\tau_i = F_i^{-1}(\Theta(v_i))$ thus $F_i(t) = P(\tau_i \leq t)$.

3.2.2 Measures of Risk

As we have defined before, the risk is the chance that an investment's actual return will be different than expected. This includes the possibility of losing some or all of the original investment. Measures of risk have a significant role in optimization under uncertainty, especially in coping with the losses that might be incurred in finance or the insurance industry.

We can predict loss as a function $z = f(x, y)$, where $x \in X \subset \mathbb{R}^n$ is a decision vector that represents the portfolio, and $y \in Y \subset \mathbb{R}^m$ is a random vector that represents future values of a number of variables like interest rates or weather data. The vector y is governed by a Borel measure \mathbb{P} on Y and is independent of x .

Two of the main popular measures of risk are the value at risk, VaR_α , and the conditional value at risk, $CVaR_\alpha$. Considering a given confidence level α , we can define VaR_α as the smallest loss of the worst α % of losses and $CVaR_\alpha$ is the average of the worst α % of losses.

More precisely, for each x we define by $\Psi(x, \cdot)$ on \mathbb{R} the resulting distribution function for the loss $z = f(x, y)$, i.e., $\Psi(x, \zeta) = \mathbb{P}\{y | f(x, y) \leq \zeta\}$. The VaR_α of the loss associated with a decision x is the value $\zeta_\alpha(x) = \min\{\zeta | \Psi(x, \zeta) \geq \alpha\}$ where the minimum is attained because $\Psi(x, \zeta)$ is non-decreasing and right-continuous in ζ . The $CVaR_\alpha$ of the loss associated with a decision x is the value $\phi_\alpha(x) \equiv$ mean of the α -tail distribution of $z = f(x, y)$ defined by:

$$\Psi_\alpha(x, \zeta) = \begin{cases} 0 & \text{for } \zeta < \zeta_\alpha(x), \\ (\Psi(x, \zeta) - \alpha)/(1 - \alpha) & \text{for } \zeta \geq \zeta_\alpha(x). \end{cases}$$

In our case, we are going to calculate VaR_α and $CVaR_\alpha$ from the loss distribution β_L :

- The α -Value at Risk:

$$VaR_\alpha(\beta_L) = \inf\{L' | \int_0^{L'} \beta_L(x) dx > (1 - \alpha)\}.$$

- The α -Conditional Value at Risk:

$$CVaR_\alpha(\beta_L) = \frac{1}{\alpha} \int_0^\alpha \inf\{L' | \int_0^{L'} \beta_L(x) dx > (1 - p)\} dp.$$

CVaR has fundamental properties as a measure of risk with significant advantages over VaR. CVaR is able to quantify dangers beyond VaR and, moreover, it is more mathematically coherent [19].

4 Portfolio Structure and Development

We apply a Default time model [17] derived from the KMV's model.

This is the general structure of the portfolio's analysis:

- Input.
- Evaluate the loss density function of the associated portfolio.
- Compute the desired portfolio performance indicators.

Step 1. Collect facilities' data:

The portfolio is formed by n firms. The bank during the portfolio's constitution set these data for each $i = 1, \dots, n$:

N_i	Nominal
T_i	Maturity date
Sp_i	Spread
$Risk_i$	Risk.

For each client associated to facility i private institutions can give us these informations:

C_i	Country of business
I_i	Industry sector
Rat_i	Rating
GRR_i	The global recovery rate, which is the recoverable amount in case of default ($GRR_i = 1 - LGD_i$)
R_i	R-square, represents the degree of correlation between the value of a client's asset and the behavior of the global economy.

In our case:

- We build each N_i randomly in $[0, 10^7]$
- The risk is a random integer number between 1 and 10, where 10 will be the highest probability of default
- The maturity T_i time will be one year
- We use the tangent hyperbolic function in order to calculate F^{-1}
- For each LGD_i we set the value 0.6
- We use an alpha value of $\alpha = 0.01$
- Finally, we obtain the correlation matrix Σ calculating a matrix of correlation coefficients for the matrix X , which is a $2nxn$ matrix being n the number of clients. Each column represents each client for $1, \dots, n$ and each row is an observation from the different values of the firm.

Step 2. Evaluate the loss density function:

We develop this program with MATLAB, and its structure is:

For j going from 1 to M. *Monte-Carlo scheme where M correspond the number of iterations*

- **Generate a Gaussian vector V:** $V = \Xi G = (v_1, \dots, v_n)$, where Ξ is the Cholesky's decomposition of the correlation matrix $\Sigma = \Xi^t \Xi$ and G is a defined standard gaussian vector
- **Generate a default time vector T:** We compute it from $\tau_i = F_i^{-1}(\Theta(v_i))$ where F^{-1} is the marginal default probability function of τ defined by $F_i(t) = P(\tau_i \leq t)$ and $(\Theta(v_i))$ is the standard normal gaussian density function
- **Compute the loss amount:** In each scenario j we calculate the loss from $L(j) = L(j) + LGD(i) * Nom(i)$ that is by taking the sum of previous losses and product of the current lost given default and the facility nominal

EndFor

Step 3. Compute the desired portfolio performance indicators:

Using data stocked in the loss' vector we can compute some indicators:

1. The discrete loss density function β_L : We discretize the possible losses' vector as $disL = \min(L) : (\max(L) - \min(L)) / (nbuck - 1) : \max(L)$ where $nbuck$ is defined depending on the stepsize, after that we obtain the histogram as $v = \text{hist}(L, disL)$ to compute the loss density function.
2. The Value at Risk (VaR): we accumulate the probability as we move along a discretized range until we reach α and obtain the corresponding loss value:

$$VaR_\alpha(\beta_L) = \inf[L' | \int_0^{L'} \beta_L(x) dx > (1 - \alpha)]$$

3. The Conditional-VaR (CVaR): We have calculated CVaR using two different methods obtaining similar results:
 - We find the average between the amounts of losses that are still over VaR

- We calculate the area of the rectangle corresponding to the function of each loss that is still over α and we divide the sum of all by α : $CVaR_\alpha(\beta_L) = \frac{1}{\alpha} \int_0^\alpha \inf[L' | \int_0^{L'} \beta_L(x) dx > (1-p)] dp$.

5 Results obtained

Our main objective is to analyze the portfolio's tendency depending on the changes of the variables. First of all, we have studied the behavior of the portfolio's losses in the general case. Afterwards, we have modified each parameter separately coming to some interesting conclusions. Finally, all parameters have been changed at the same time.

5.1 Disturbing Client and the model iteration number

We launch the program for different client's numbers and Montecarlo's iterations. For each combination we launch the program ten times and we make its mean. Thus, we obtained these results:

		Number of Clients			
		100	250	500	1000
1000 Iterations	Total Nominal	5,248E+08	1,222E+09	2,471E+09	4,813E+09
	VaR	6,825E+07	1,386E+08	2,385E+08	4,411E+08
	% vs Total Nominal	13,01%	11,34%	9,65%	9,16%
	CVaR	7,173E+07	1,438E+08	2,455E+08	4,479E+08
	% vs Total Nominal	13,67%	11,76%	9,93%	9,31%
5000 Iterations	VaR	6,931E+07	1,404E+08	2,381E+08	4,399E+08
	% vs Total Nominal	13,21%	11,49%	9,64%	9,14%
	CVaR	7,325E+07	1,460E+08	2,460E+08	4,510E+08
	% vs Total Nominal	13,96%	11,95%	9,95%	9,37%
10000 Iterations	VaR	6,968E+07	1,403E+08	2,387E+08	4,405E+08
	% vs Total Nominal	13,28%	11,48%	9,66%	9,15%
	CVaR	7,377E+07	1,460E+08	2,459E+08	4,508E+08
	% vs Total Nominal	14,06%	11,95%	9,95%	9,37%
100000 Iterations	VaR	6,955E+07	1,402E+08	2,388E+08	4,417E+08
	% vs Total Nominal	13,25%	11,47%	9,66%	9,18%
	CVaR	7,376E+07	1,462E+08	2,467E+08	4,528E+08
	% vs Total Nominal	14,06%	11,96%	9,98%	9,41%

Table 1: Disturbing clients and Montecarlo's iterations.

If we consider enough Montecarlo's iteration, which means a large number of possible scenarios, we can observe that the balanced sum of the amount of losses approximately distributes as a normal variable. In probability theory, the central limit theorem, developed by Aleksandr Lyapunov (1901) [8], states conditions under which the mean of a sufficiently large number of independent random variables, each with finite mean and variance, will be approximately normally distributed.

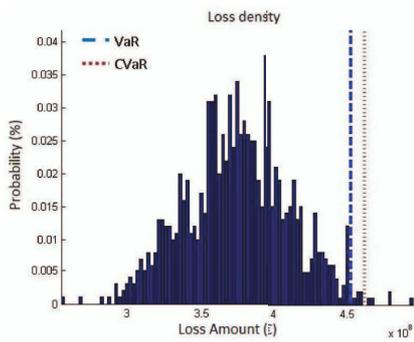


Figure 3: 1000 clients running with 1000 iterations.

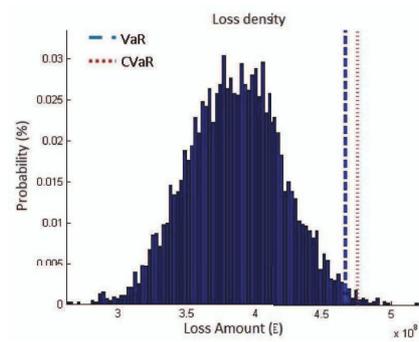


Figure 4: 1000 clients running with 5000 iterations.

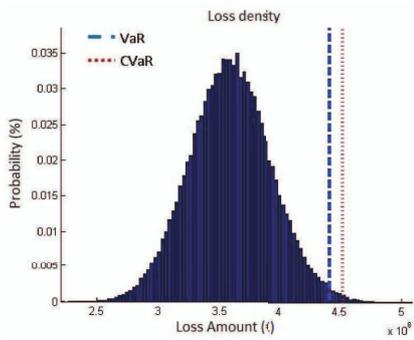


Figure 5: 1000 clients running with 10000 iterations.

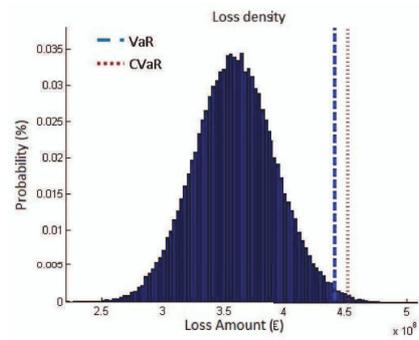


Figure 6: 1000 clients running with 100000 iterations.

The graph of the associated probability density function is bell-shaped, with a peak at the mean, and is known as the Gaussian function or bell curve. Obviously, as the number of iterations increases, the obtained results will be more accurate, as we can observe in the graphics.

Generally, we have noticed that there exists an inversely proportional relationship between the percentage of losses and the number of MonteCarlo's iterations. As more scenarios are considered, the risk measures VaR and

CVaR decreases. This is due to the fact that, as it has been said before, the amount of losses distributes as a normal variable so the values tend to cluster around the mean. Consequently, following the definition of VaR and CVaR, both measures will take a lower value. We can see in the figure 7 a graphic showing how the VaR and CVaR change as the Montecarlo's iteration are risen.

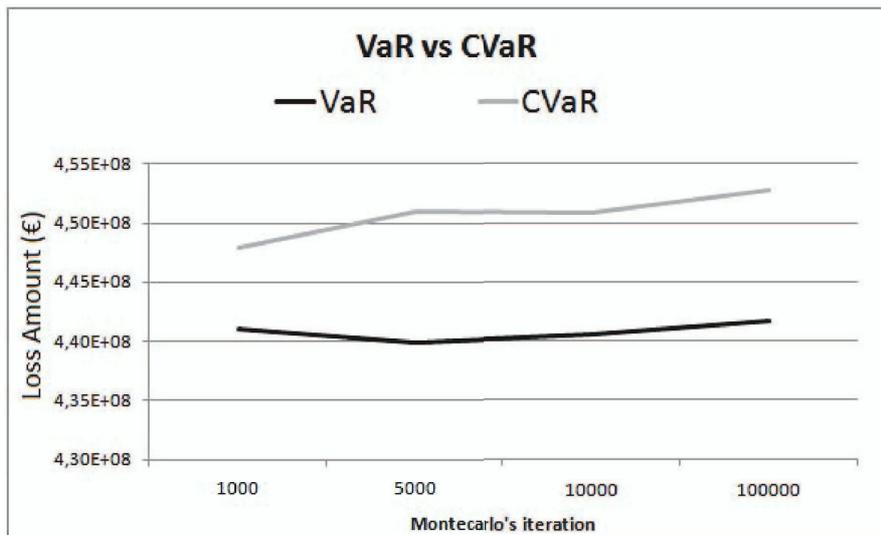


Figure 7: VaR vs CVaR. Make for 1000 clients.

It also has to be mentioned the comparative between CVaR and the nominal. Usually, the percentage of losses takes values between 15% and 9%. As more clients are considered, this percentage decreases and it approaches to the 9%. It is worth noting that diversifier the portfolio is, better results are obtained because of risk decrease.

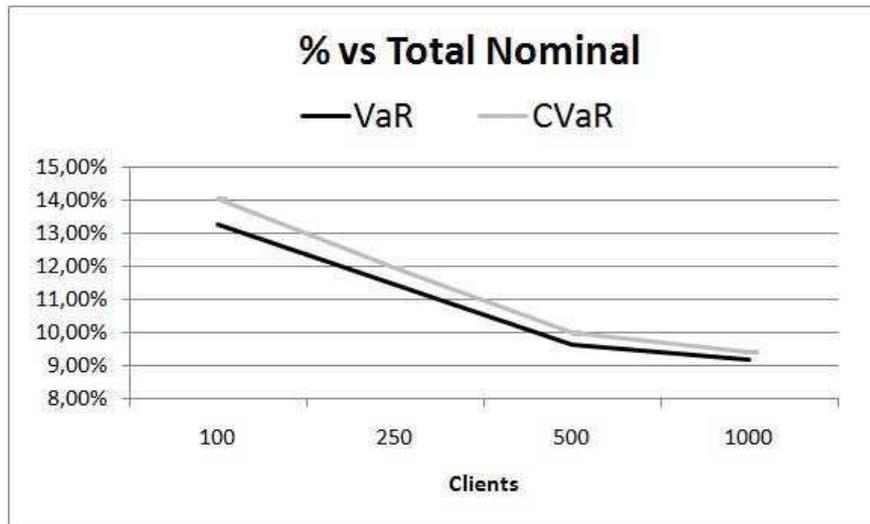


Figure 8: Var vs Total Nominal. Make for 100000 Montecarlo's iteration.

Only positive correlation

Finally, as we have mentioned before, in some literature only positive correlation is considered because its likelihood to reality. Besides we will work with both positive and negative values, we have launched the program with just positive correlation to analyze it briefly. We can appreciate that VaR and CVaR decrease, which means that the final amount of losses will be a lower value. In fact, both risk measures are between a 50% and 60% lower in positives correlations than in positives and negatives correlations.

		Number of Clients			
		100	250	500	1000
Total Nominal		4,898E+08	1,264E+09	2,557E+09	5,004E+09
<u>1000</u> <u>Iterations</u>	VaR	3,127E+07	6,585E+07	1,065E+08	1,767E+08
	% vs Total Nominal	6,38%	5,21%	4,17%	3,53%
	CVaR	3,329E+07	6,830E+07	1,102E+08	1,828E+08
	% vs Total Nominal	6,80%	5,40%	4,31%	3,65%
<u>5000</u> <u>Iterations</u>	VaR	3,153E+07	6,619E+07	1,083E+08	1,767E+08
	% vs Total Nominal	6,44%	5,24%	4,24%	3,53%
	CVaR	3,436E+07	7,040E+07	1,133E+08	1,831E+08
	% vs Total Nominal	7,02%	5,57%	4,43%	3,66%
<u>10000</u> <u>Iterations</u>	VaR	3,148E+07	6,636E+07	1,078E+08	1,769E+08
	% vs Total Nominal	6,43%	5,25%	4,22%	3,53%
	CVaR	3,415E+07	7,071E+07	1,134E+08	1,836E+08
	% vs Total Nominal	6,97%	5,59%	4,43%	3,67%
<u>100000</u> <u>Iterations</u>	VaR	3,155E+07	6,610E+07	1,077E+08	1,766E+08
	% vs Total Nominal	6,44%	5,23%	4,21%	3,53%
	CVaR	3,442E+07	7,039E+07	1,132E+08	1,839E+08
	% vs Total Nominal	7,03%	5,57%	4,43%	3,68%

Table 2: Results only for positive correlation.

5.2 Disturbances in the program

Before starting to analyze the results obtained, we would like to mention that despite taking the disturbances as independent, all the parameters in our algorithm are connected. So, besides having studied them separately, the results will be affected by the rest of variables. However, we have tried to fix the rest of the components as much as we could so the conclusions would not be that affected.

EXAMPLE 1			EXAMPLE 2		
	Total Nominal	2,51E+09		Total Nominal	2,45E+09
	VaR	2,44E+08		VaR	2,47E+08
	% vs Total Nominal	9,70%		% vs Total Nominal	10,07%
	CVaR	2,50E+08		CVaR	2,54E+08
	% vs Total Nominal	9,95%		% vs Total Nominal	10,37%
	Norm	1,6946		Norm	1,6899
<u>Slope's</u> <u>function</u> <u>of risk</u>	<u>10</u>	33,10%	<u>Slope's</u> <u>function</u> <u>of risk</u>	<u>10</u>	33,10%
	<u>9</u>	24,90%		<u>9</u>	24,90%
	<u>8</u>	19,95%		<u>8</u>	19,95%
	<u>7</u>	16,64%		<u>7</u>	16,64%
	<u>6</u>	14,27%		<u>6</u>	14,27%
	<u>5</u>	12,49%		<u>5</u>	12,49%
	<u>4</u>	11,10%		<u>4</u>	11,10%
	<u>3</u>	9,99%		<u>3</u>	9,99%
	<u>2</u>	9,09%		<u>2</u>	9,09%
	<u>1</u>	8,33%		<u>1</u>	8,33%
<u>Client's</u> <u>with</u> <u>default</u> <u>risk</u>	<u>10</u>	51	<u>Client's</u> <u>with</u> <u>default</u> <u>risk</u>	<u>10</u>	54
	<u>9</u>	49		<u>9</u>	45
	<u>8</u>	58		<u>8</u>	50
	<u>7</u>	52		<u>7</u>	65
	<u>6</u>	47		<u>6</u>	41
	<u>5</u>	55		<u>5</u>	45
	<u>4</u>	40		<u>4</u>	57
	<u>3</u>	43		<u>3</u>	47
	<u>2</u>	59		<u>2</u>	48
	<u>1</u>	46		<u>1</u>	48

Table 3: Original data from each example.

We have considered 500 clients and 5000 Montecarlo's iterations in every disturbance. The Table 3 correspond with the original data that we have later worked with. Consequently, two examples have been made in order to obtain two different results.

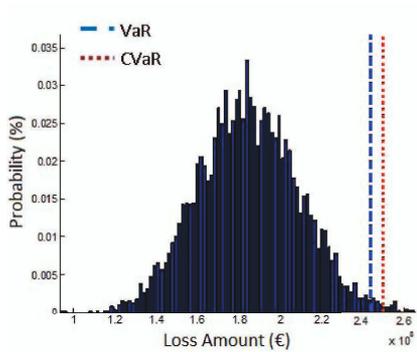


Figure 9: Loss density function in Example 1.

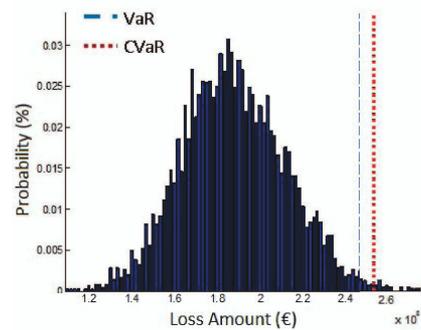


Figure 10: Loss density function in Example 2.

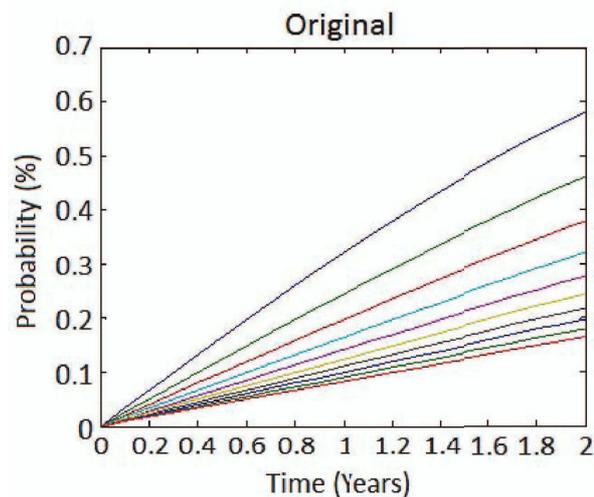


Figure 11: Each function in the image represents a different risk level, with the highest function corresponding to risk level 10, the highest, and with the lowest function corresponding to risk level 1, the lowest.

5.2.1 Nominal

First of all, we have modified the amount of money which the portfolio is made up of. Obviously, it exists a basic relationship between the nominal invested and the final amount of losses. But as we are working with percentages, we will study if the VaR and CVaR vary when modifying the nominal.

The principal problem when talking about nominal is that it really depends on the associated risk. In each simulation, the results are influenced by the risk related to the firm which nominal is been disturbed. Higher risk

represents large amount of losses, besides lower risks means less probability of default. Therefore, if we reduced the amount of money related to firms with high probability of default or increase the one associated with low risk, then VaR and CVaR values will fall. On the other hand, VaR and CVaR will take higher values if either the nominal associated with a low risk decreases or if we invest more money in a firm with a high percentage of default.

The table 4 is the result of disturbing randomly a ± 10 percentage of each client nominal.

		Case number										Average	Standard deviation
% vs		1	2	3	4	5	6	7	8	9	10		
Total Nom	Original	-0,04%	0,23%	0,00%	-0,08%	0,76%	0,42%	0,02%	0,29%	0,09%	0,01%	0,20%	0,24%
VaR	Original	0,24%	0,27%	-0,42%	-0,05%	0,84%	0,60%	0,34%	-0,40%	-0,31%	-0,55%	0,40%	0,22%
	Total Nom	9,73%	9,73%	9,66%	9,70%	9,79%	9,76%	9,74%	9,67%	9,67%	9,65%	9,71%	0,05%
Cvar	Original	0,47%	0,44%	-0,30%	0,02%	0,68%	0,50%	0,44%	-0,33%	0,12%	-0,42%	0,37%	0,19%
	Total Nom	9,99%	9,99%	9,92%	9,95%	10,01%	10,00%	9,99%	9,91%	9,96%	9,90%	9,96%	0,04%

Table 4: Nominal's disturbances for Example 1.

The fact that the nominal can increase or decrease vs the original amount does not mean that both risk measures have to do it at the same time. If we consider the results obtained in the case number 1, the nominal is 0,04% lower than the beginning while VaR and CVaR increase 0,24% and 0,47% respectively. On the other hand in the fourth column, VaR decreases a 0,05% while the nominal is $-0,08%$ vs Original. The case in which the nominal is increase can be described with analogy. Moreover, the third case shows no final change in the nominal while VaR and CVaR decrease significantly.

The figure 12 has been inserted as another way to show that same percentage of disturbed nominal can give different results in VaR.

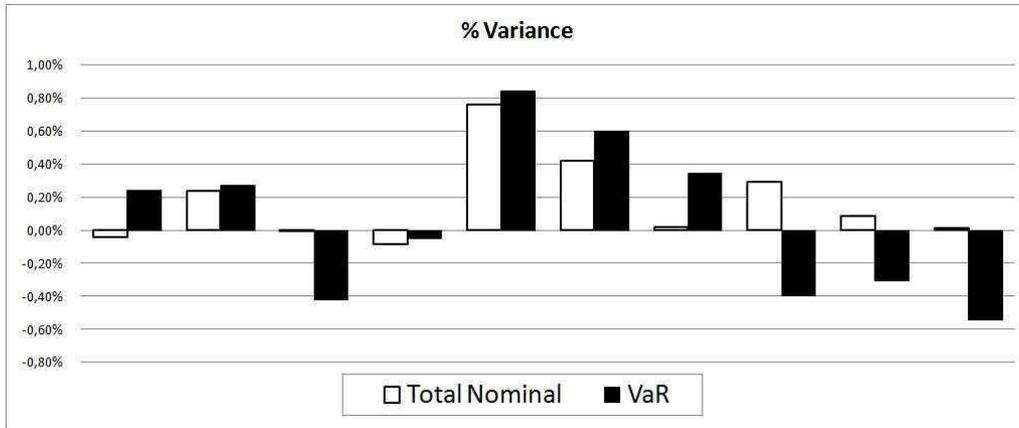


Figure 12: Nominal vs VaR.

However, if we modified the nominal significantly we will obtain the obvious results: lower nominal means less amount of losses and vice versa. The table 5 represents the percentage of VaR and CVaR vs the original one modifying the nominal as much as the first row mentions to the 10% of the firms.

	Nomina's disturbance									
	-80%	-60%	-40%	-20%	-10%	10%	20%	40%	60%	80%
VAR	7,05E+07	6,83E+07	6,07E+07	6,59E+07	6,56E+07	6,51E+07	6,40E+07	6,35E+07	6,29E+07	6,14E+07
%	7,91%	4,55%	2,51%	0,84%	0,42%	-0,42%	-2,06%	-2,89%	-3,81%	-6,01%
CVAR	7,49E+07	7,27E+07	7,12E+07	6,97E+07	6,92E+07	6,85E+07	6,72E+07	6,66E+07	6,58E+07	6,48E+07
%	8,76%	5,60%	3,36%	1,26%	0,52%	-0,55%	-2,35%	-3,27%	-4,36%	-5,88%

Table 5: Nominal's disturbances.

5.2.2 Risk

As we were developing the algorithm and analyzing the results obtained with the disturbances in the portfolio, we have come to the conclusion that the risk is the most sensitive parameter in the portfolio. Evidently, the losses increase as the risk takes higher values. But also, VaR and CVaR change significantly their percentage depending on which risks are altered.

We have modified ± 1 the risk value of the 10 percent of the total clients:

		Case number										Average	Standard deviation
		1	2	3	4	5	6	7	8	9	10		
Client's with default risk	10	-1	+3	-1	+0	-8	-2	-10	-3	-3	-4	-3	2,8
	9	+1	-5	+0	+3	+9	+2	+9	+3	+5	+6	+3	3,5
	8	+1	+3	+2	-3	-3	+7	+0	+4	-3	-2	-0	3,0
	7	+2	+1	+1	-4	-1	-12	+2	-3	+0	+3	-1	3,3
	6	-1	-7	+2	+4	+1	+3	+2	-2	+3	-1	+1	2,5
	5	-4	+0	-6	-4	+2	+0	-6	+0	+0	-2	-1	2,5
	4	+0	+7	+0	+4	+1	+0	+6	+0	-3	+0	+1	2,8
	3	+2	+0	+4	-1	+0	+3	-1	+0	+1	+1	+1	2,2
	2	+3	+2	+2	+4	+0	+3	-1	+8	+0	-2	+2	4,6
	1	-3	-4	-4	-3	-1	-4	-1	-7	+0	+1	-3	3,3
% vs													
VaR	Original	0,11%	0,27%	-0,24%	0,09%	-1,36%	-0,51%	-1,65%	-0,30%	-0,98%	-1,25%	0,68%	0,58%
	Total Nom	9,72%	9,73%	9,68%	9,71%	9,57%	9,65%	9,54%	9,68%	9,61%	9,58%	9,64%	0,05%
% vs													
CVaR	Original	0,17%	0,21%	-0,17%	0,33%	-1,15%	-0,66%	-1,11%	-0,14%	-1,02%	-1,28%	0,62%	0,48%
	Total Nom	9,96%	9,97%	9,93%	9,98%	9,83%	9,88%	9,84%	9,93%	9,84%	9,82%	9,89%	0,05%

Table 6: Risk's disturbances for Example 1.

Each number that appears in the row client's with default risk and column case number matches with the counter that varies depending on the arrival or departure of clients to the risk. For example, in the first column, if we take a look to the firms with risk 5, we will assume that 4 clients have a lower risk than before the disturbance. Risks 10 and 1 will only be able to decrease and increase respectively because of the non existence of risks 11 and 0. We have created these counters as a consequence of the importance of which risks and how they are modified. However, a counter considering all the disturbances at the same time might not be useful. Total risk counters are equal in the cases 2 and 6 while their percentages of losses vs. the original are not even the same sign. This is because the influence of the high risk on the amount of losses is much more significant than the lower ones. For example, at case 7, clients with default risk 6, 7, 8 and 9 are increased, but VaR and CVaR have the strongest decrease. This is due to the fact that clients with the highest risk are decreased in 10 firms. We also have to consider the relation between nominal and risk, as there exists an important dependence. In the cases 1 and 3, the disturbance in the risk is similar, but VaR and CVaR are completely different. This is due either to high or low nominal associated with risk 10 and risks 9 and 8 respectively.

According to the table 6, we can observe that despite of the fact that two scenarios have the same modification (talking about numbers) in the risk, VaR or CVaR are not equal. The amount of losses is much more influenced

by higher values than by lower risks, which in our opinion seems reasonable if we take a look to the figure 13 of the survival function.

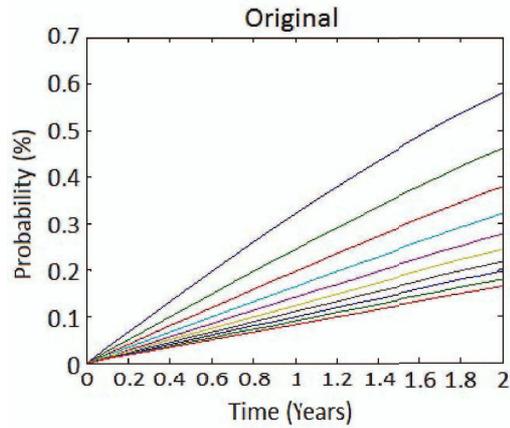


Figure 13: Survival functions for each disturbance.

The main explanation is that differences in the probability of default of the firms with distinct risks are localized in the ones with a high percentage of arriving to the default time before the maturity time. In addition, as we have showed in the table above, VaR and CVaR are always positive. This fact is also caused by the increase in the amount of losses of the firms with high risks, whose sum is much higher than the amount of money not defaulting obtained by the decrease of the risk value.

		Case number									
% vs		Risk 1	Risk 2	Risk 3	Risk 4	Risk 5	Risk 6	Risk 7	Risk 8	Risk 9	Risk 10
VaR	VaR	1,26E+08	1,36E+08	1,50E+08	1,68E+08	1,85E+08	2,12E+08	2,47E+08	2,97E+08	3,73E+08	5,01E+08
	First	0,00%	7,41%	18,96%	32,89%	46,28%	67,77%	95,24%	135,01%	195,19%	296,21%
	Previous	0,00%	7,41%	10,76%	11,71%	10,08%	14,69%	16,37%	20,37%	25,51%	34,22%
CVaR	CVaR	1,32E+08	1,42E+08	1,56E+08	1,73E+08	1,92E+08	2,19E+08	2,53E+08	3,04E+08	3,82E+08	5,09E+08
	First	0,00%	7,38%	18,32%	31,11%	45,10%	65,58%	91,77%	130,47%	189,42%	285,36%
	Previous	0,00%	7,38%	10,19%	10,81%	10,67%	14,11%	15,81%	20,18%	25,58%	33,15%

Table 7: Risk's disturbances for Example 1.

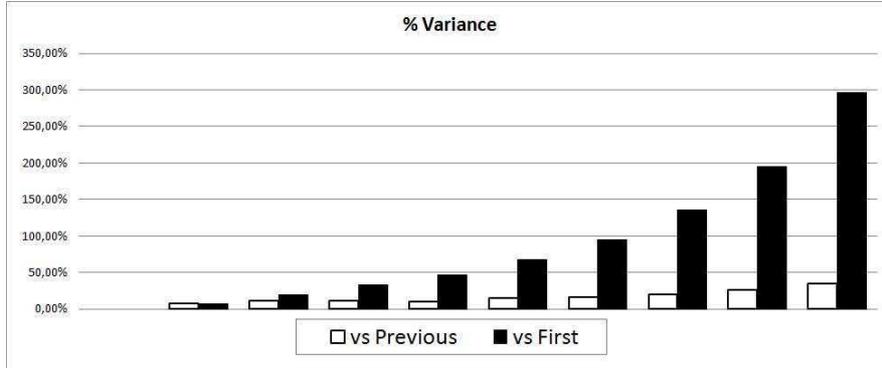


Figure 14: % VaR variance.

We confirm this conclusion in the table 7 which has been the result of setting the nominal and making the risk being the same for all firms each time. From this table, we have created the figure 14, that illustrates the increase of VaR vs Risk value.

Finally, the table 8 shows the behavior of VaR and CVaR depending of the risk. We have modified each client's risk from +5 to -5, and we can observe that the absolute value of the percentage of VaR is higher when decreasing the risk value.

	Risk's disturbance									
	5	4	3	2	1	-1	-2	-3	-4	-5
VAR	6,31E+07	6,33E+07	6,40E+07	6,41E+07	6,46E+07	6,62E+07	6,63E+07	6,71E+07	6,76E+07	6,82E+07
%	-3,52%	-3,21%	-2,14%	-1,99%	-1,22%	1,22%	1,38%	2,60%	3,36%	4,28%
CVAR	6,61E+07	6,64E+07	6,70E+07	6,73E+07	6,76E+07	6,98E+07	7,00E+07	7,08E+07	7,14E+07	7,20E+07
%	-3,99%	-3,56%	-2,69%	-2,25%	-1,82%	1,38%	1,67%	2,83%	3,70%	4,58%

Table 8: Risk's disturbances.

5.2.3 Correlation Matrix

The correlation with clients will be modified from -10% to 10% of the previous correlation. The disturbance in the firm's correlation does not give any significant information. The decrease or increase in VaR and CVaR are not due to the disturbance in the matrix. While the norms of the X matrix case 2 and 3 in the table 9 are 0.02% and -0.01% ; VaR values are practically equal. Moreover the values of the percentages of the measures of Var vs the original ones are not as high as usual. We might think that in case 8 that the difference in the actual VaR and the original is really big, but we have to take into account the range of values. In fact it is just a -0.52% .

		Case number											
% vs		1	2	3	4	5	6	7	8	9	10	Average	Standard deviation
Norm of the X Matrix	Original	0,00%	0,02%	-0,01%	0,01%	-0,01%	-0,02%	0,04%	0,04%	0,01%	0,01%	0,01%	0,01%
	Total Nom	9,70%	9,69%	9,69%	9,70%	9,68%	9,73%	9,72%	9,65%	9,71%	9,74%	9,70%	0,02%
VaR	Original	-0,01%	-0,14%	-0,15%	-0,09%	-0,22%	0,22%	0,13%	-0,52%	0,04%	0,35%	0,19%	0,15%
	Total Nom	9,70%	9,69%	9,69%	9,70%	9,68%	9,73%	9,72%	9,65%	9,71%	9,74%	9,70%	0,02%
CVaR	Original	-0,18%	-0,13%	-0,06%	-0,22%	0,12%	-0,10%	0,02%	-0,78%	-0,13%	0,00%	0,17%	0,22%
	Total Nom	9,93%	9,93%	9,94%	9,92%	9,96%	9,94%	9,95%	9,87%	9,93%	9,95%	9,93%	0,02%

Table 9: Matrix's disturbances for Example 1.

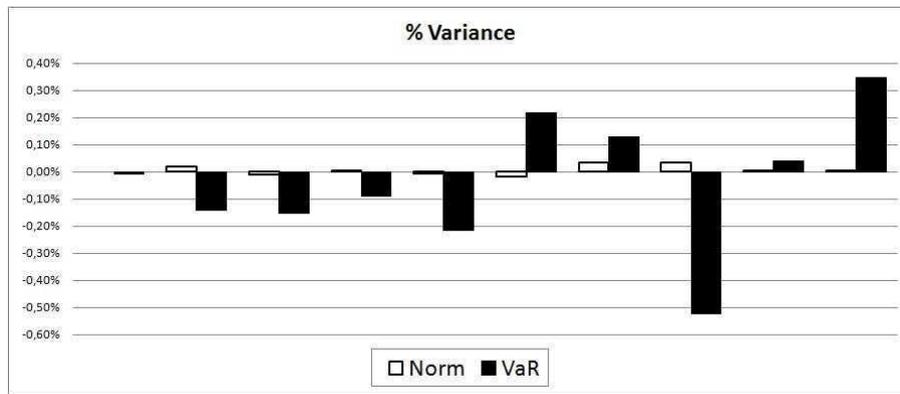


Figure 15: Norm vs VaR.

As it is explained above, in order to obtain the times of default, we applied the Cholesky triangle that consists in the decomposition of a symmetric and positive-definite matrix into the product of a lower triangular matrix and its conjugate transpose. However, in our algorithm we calculate the default correlation matrix by creating a matrix composed with scalar values drawn from a normal distribution with mean 0 and standard deviation 1. This means that the matrix might not be positive-definite in every case and consequently the Cholesky decomposition will not be done.

5.2.4 Survival function

In order to modify the time values obtained with the survival function, we had to be specially careful keeping each risk with a suitable range of values. We have modified each survival function's slope in a ± 10 percent in each case and we obtained these results:

		Case number										Average	Standard deviation
		1	2	3	4	5	6	7	8	9	10		
Slope's function of risk	10	-0,65%	2,16%	2,93%	1,15%	-2,16%	-0,08%	1,50%	1,78%	2,71%	-3,03%	1,82%	0,98%
	9	2,31%	2,18%	1,96%	0,25%	0,28%	-2,35%	0,28%	1,90%	-2,38%	-0,05%	1,39%	1,03%
	8	1,74%	0,37%	-1,99%	-0,31%	-1,92%	-0,59%	-0,51%	1,60%	-0,61%	-0,26%	0,99%	0,72%
	7	0,59%	0,12%	-1,22%	1,41%	-0,26%	-0,38%	-0,12%	-1,30%	0,13%	-1,65%	0,72%	0,61%
	6	1,14%	0,85%	-0,33%	1,26%	0,76%	0,08%	0,32%	1,33%	-1,04%	-0,78%	0,79%	0,43%
	5	0,82%	-0,93%	1,18%	0,49%	-0,78%	-0,20%	-0,94%	-0,85%	0,59%	0,46%	0,72%	0,29%
	4	0,29%	-0,22%	0,40%	-0,05%	0,11%	-0,91%	0,45%	-0,01%	-0,72%	0,98%	0,41%	0,35%
	3	0,92%	0,96%	0,08%	-0,12%	0,50%	0,25%	0,20%	0,78%	0,35%	0,91%	0,51%	0,35%
	2	-0,38%	-0,73%	0,72%	0,47%	-0,83%	0,49%	-0,04%	0,01%	-0,81%	-0,57%	0,51%	0,29%
	1	-0,24%	0,08%	-0,12%	-0,69%	0,59%	0,13%	0,73%	-0,17%	0,43%	0,15%	0,33%	0,25%
Average		0,65%	0,48%	0,36%	0,39%	-0,37%	-0,36%	0,19%	0,51%	-0,14%	-0,38%	0,38%	0,15%
% vs													
VaR	Original	4,79%	3,27%	2,91%	3,00%	-2,66%	-2,28%	0,86%	3,46%	-1,34%	-3,99%	2,86%	1,17%
	Total Nom	10,17%	10,02%	9,99%	9,99%	9,45%	9,48%	9,79%	10,04%	9,57%	9,32%	9,78%	0,30%
% vs													
CVaR	Original	4,85%	3,57%	2,91%	3,08%	-2,89%	-2,23%	1,25%	3,73%	-0,62%	-4,20%	2,93%	1,29%
	Total Nom	10,43%	10,30%	10,24%	10,25%	9,66%	9,72%	10,07%	10,32%	9,89%	9,53%	10,04%	0,32%

Table 10: Survival's function's disturbances of the Example 1.

It could be observed, in the table 10, that in this disturbance a negative average in the sum of all slope's functions of risk defines a decrease in the risk measures. We really can appreciate it observing the figure 16. In this case, if we decrease the survival probability function, there then a large number of firms will default before maturity time and the majority of firms which default will do it with a loss close to the mean. Therefore risk measures VaR and CVaR decrease. For example if we take a look not only to this table, but also to the graphs added in the figure 17, we can appreciate that the survival function of higher risks have been decrease the most and, in fact, VaR and CVaR percents are the lowest ($-3,99\%$ and $-4,20\%$). This is what we have been explained before. Nevertheless, the biggest disturbance in the risk measures is produced by modifications in the slope. But we will deeply study these conclusions afterwards. Talking about positive changes in VaR and CVaR percentage versus original, it is worth mentioning cases 2, 4 and 8 which most increase the survival function related with risks 9 and 10. We also can check this fact in the figures mentioned before.

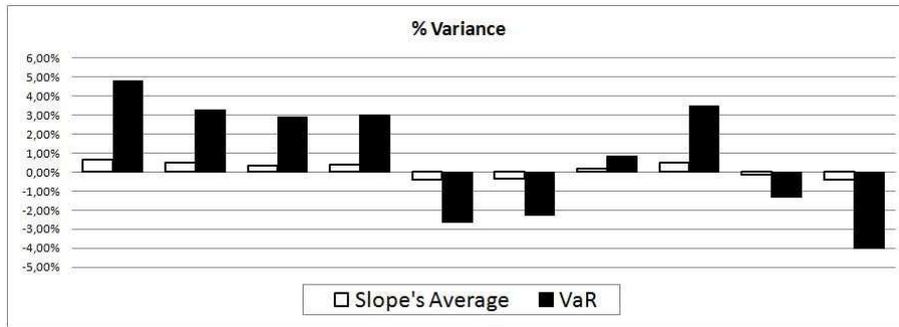


Figure 16: Survival's function vs VaR.

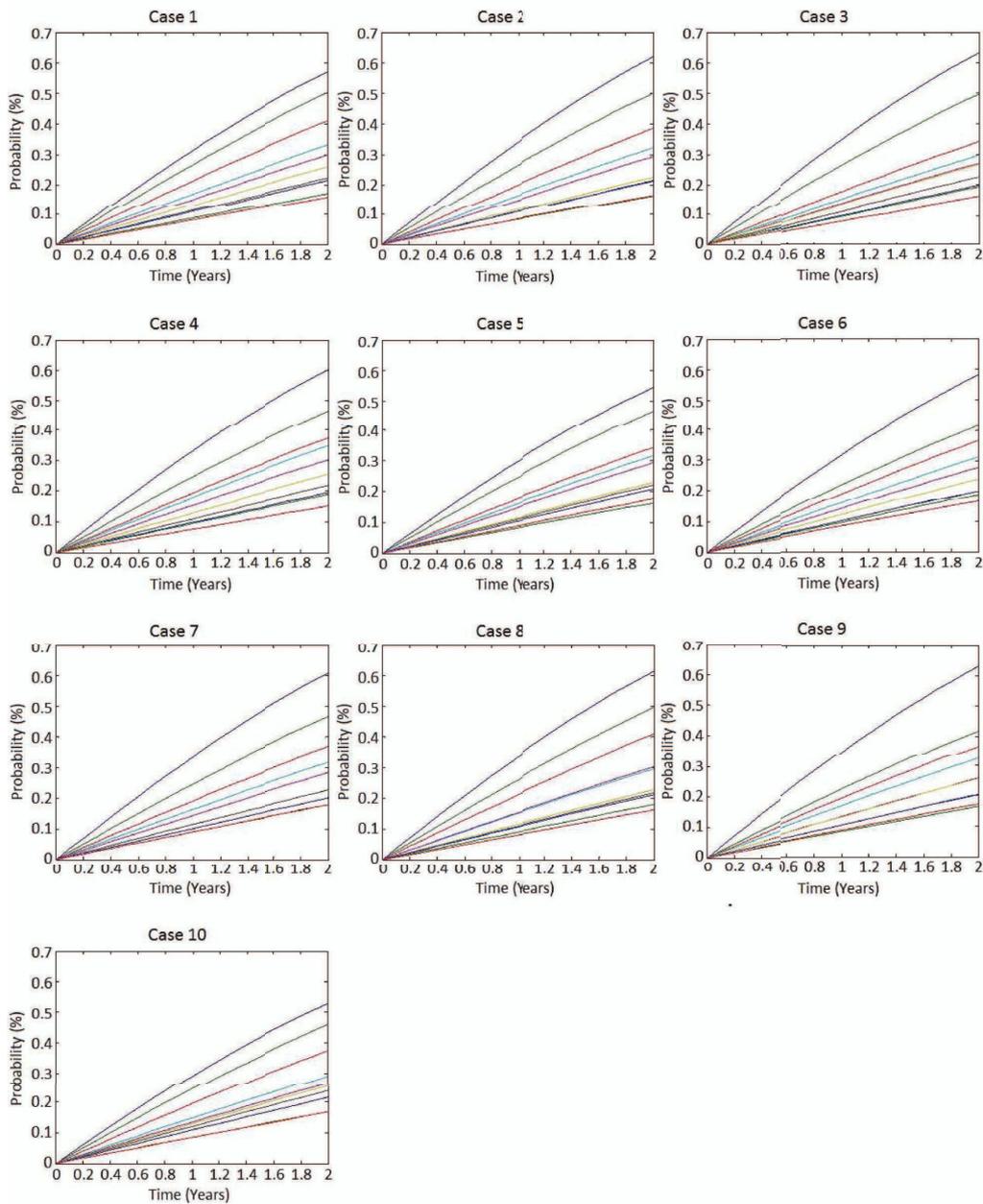


Figure 17: Survival's function for each case.

5.2.5 All together

To sum up, we have modified all the parameters in the way explained before in order to have a more realistic data.

Moreover, we show both examples in order to affirm our conclusions. First of all, let's focus on the example number 1. If we observe the figure 18, which represents the percentage of variance, we can notice that the biggest bars related to VaR match with the slopes function of risks disturbance. Cases number 1 and 9 show similar degrees in this risk measure (-2.9% and -2.95%) while both total nominal and norm of the X matrix percentage have even different sign. However in both cases the average of the slopes function are highly decreased. On the other hand cases from 4 to 6 have analogous results. Apart from the disturbance in the survival function, the nominal has also an important role. The main difference between case 7 and 8 relies on the nominal (0.37% and -0.41% respectively) because the rest of the parameters are quite similar.

		Case number										Average	Standard deviation
		1	2	3	4	5	6	7	8	9	10		
Total Nominal	% vs Original	0,09%	-0,21%	0,06%	0,06%	0,01%	0,16%	0,37%	-0,41%	-0,22%	-0,17%	0,18%	0,13%
Slope's function of risk	10	0,55%	1,15%	2,68%	2,19%	2,41%	2,31%	1,95%	1,41%	-0,06%	1,88%	1,66%	0,85%
	9	-2,17%	0,22%	1,39%	-0,17%	-0,13%	2,01%	-1,07%	1,18%	1,13%	-1,50%	1,10%	0,73%
	8	-1,99%	0,18%	-0,89%	1,13%	0,85%	0,33%	1,06%	0,98%	0,41%	-1,19%	0,90%	0,52%
	7	-1,00%	-0,46%	-1,35%	0,91%	0,07%	-0,25%	-0,17%	-0,43%	-1,55%	0,85%	0,70%	0,51%
	6	-0,75%	1,09%	0,10%	-0,44%	0,38%	0,65%	0,99%	1,09%	-0,72%	-0,13%	0,63%	0,37%
	5	1,15%	-0,57%	1,04%	-0,48%	-0,78%	-1,08%	-1,21%	-0,71%	-1,18%	0,65%	0,89%	0,28%
	4	-0,37%	0,00%	-0,15%	0,98%	0,86%	-0,32%	0,99%	-0,61%	-0,77%	-0,26%	0,53%	0,36%
	3	-0,37%	-0,67%	-0,28%	0,78%	-0,35%	-0,62%	0,35%	0,18%	-0,93%	0,03%	0,46%	0,28%
	2	-0,40%	0,24%	-0,38%	-0,37%	-0,25%	0,01%	0,49%	-0,11%	0,16%	-0,43%	0,28%	0,16%
	1	0,19%	0,02%	-0,81%	0,33%	-0,29%	-0,02%	-0,23%	-0,03%	-0,37%	0,35%	0,26%	0,24%
Average		-0,52%	0,12%	0,14%	0,49%	0,28%	0,30%	0,32%	0,30%	-0,39%	0,02%	0,29%	0,16%
Client's with default risk	10	+0	-3	-2	-2	+1	-3	+1	+0	-6	-6	-2	2,6
	9	-4	+1	+1	+1	-3	+4	+0	-3	+6	+3	+1	3,2
	8	+6	-2	+2	+2	-1	+2	+0	+8	-1	+3	+2	3,2
	7	-7	+3	-4	-4	+2	-3	-4	-3	+1	-4	-2	3,2
	6	+4	+2	+2	+2	-1	-1	+2	-2	-3	+5	+1	2,6
	5	+3	-4	-4	-4	+5	-1	+1	+0	+3	-4	-1	3,4
	4	-5	+6	+3	+3	-5	+0	+1	-3	-1	+4	+0	3,8
	3	+5	-3	+3	+3	+4	+1	+0	+2	+1	-3	+1	2,7
	2	+5	-1	-1	-1	+4	+1	-1	+0	-2	+8	+1	3,3
	1	-7	+1	+0	+0	-6	+0	+0	+1	+2	-6	-2	3,4
% vs													
Norm of the X Matrix	Original	-0,04%	-0,02%	-0,02%	0,02%	-0,02%	0,01%	0,04%	-0,01%	0,01%	-0,04%	0,02%	0,01%
VaR	Original	-2,93%	1,24%	1,11%	2,39%	1,99%	2,16%	1,32%	0,75%	-2,91%	-0,10%	1,69%	0,94%
	Total Nom	9,42%	9,82%	9,81%	9,94%	9,90%	9,91%	9,83%	9,78%	9,42%	9,69%	9,75%	0,19%
CVaR	Original	-3,19%	1,12%	1,44%	2,47%	2,03%	2,13%	1,68%	1,40%	-3,05%	0,20%	1,87%	0,91%
	Total Nom	9,63%	10,06%	10,09%	10,19%	10,15%	10,16%	10,11%	10,09%	9,64%	9,97%	10,01%	0,21%

Table 11: All together's disturbances of the Example 1.

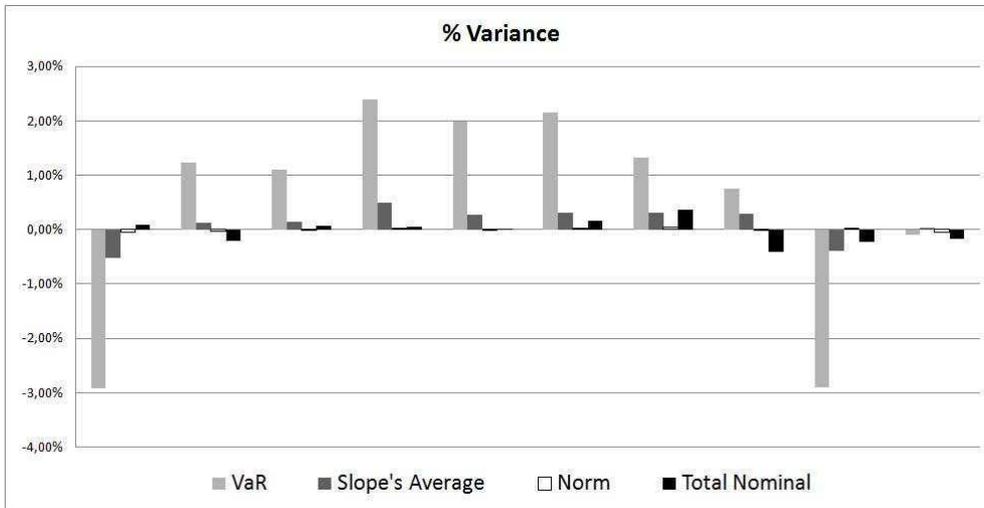


Figure 18: VaR vs Norm vs Nominal vs Survival's function of the Example 1.

In the second example, what really attracts out attention is the 9th case where the VaR percentage has increased the most. This is due to the growth of the slope's function of risk 9 and 10 (2.97% and 2.37%). Furthermore, only when the average of the slope's function is negative, the bar is under the zero-line. Another interesting case is case number 7. Despite the slope's function of risk is almost insignificant, as well as the norm of the X matrix, VaR is increased considerably. This is not only because of the increase in the nominal (which with a 0.45% is the highest), but also because of the decrease in the number of clients with risk 10 (6 firms fewer).

		Case number										Average	Standard deviation
		1	2	3	4	5	6	7	8	9	10		
Total Nominal	% vs Original	-0,02%	0,20%	-0,31%	0,18%	-0,15%	0,11%	0,45%	-0,18%	-0,12%	0,05%	0,18%	0,12%
Slope's function of risk	10	-0,63%	-2,18%	2,53%	2,40%	1,83%	-0,81%	1,25%	0,19%	2,97%	2,52%	1,73%	0,95%
	9	1,42%	2,29%	-0,45%	-0,17%	-1,04%	1,42%	2,19%	2,12%	2,37%	-1,79%	1,53%	0,78%
	8	-1,88%	1,81%	-1,64%	-1,22%	0,56%	0,98%	0,07%	-0,86%	0,70%	1,61%	1,13%	0,60%
	7	-1,02%	-1,03%	-0,18%	-1,04%	0,96%	1,13%	1,46%	0,96%	-0,66%	0,63%	0,91%	0,35%
	6	0,86%	0,96%	-1,32%	0,38%	-0,53%	0,79%	0,71%	0,48%	-1,41%	-0,83%	0,83%	0,34%
	5	1,09%	-0,53%	-0,17%	0,45%	-0,18%	0,13%	-1,05%	-0,12%	1,24%	-0,74%	0,57%	0,43%
	4	0,97%	0,10%	0,83%	0,06%	0,25%	-1,06%	-0,71%	-0,24%	1,01%	-0,77%	0,60%	0,39%
	3	-0,80%	0,56%	-0,35%	-0,70%	0,37%	-0,93%	-0,75%	-0,14%	0,97%	0,18%	0,58%	0,30%
	2	-0,06%	0,30%	0,64%	-0,45%	-0,09%	-0,85%	-0,61%	-0,80%	0,63%	0,62%	0,51%	0,27%
	1	-0,10%	-0,33%	-0,21%	-0,78%	0,06%	0,43%	0,12%	-0,53%	0,40%	-0,82%	0,38%	0,27%
Average		-0,02%	0,19%	-0,03%	-0,11%	0,22%	0,12%	-0,02%	0,11%	0,82%	0,06%	0,17%	0,24%
Client's with default risk	10	-1	-5	-3	-3	-2	+4	-6	-1	-1	-1	-2	2,7
	9	+3	+7	-1	+2	+1	-6	+7	+2	+3	+1	+2	3,8
	8	-3	-3	+5	-3	+0	+3	+0	+5	-3	+0	+0	3,2
	7	-4	+0	-5	+0	+1	+1	-1	-1	-5	-2	-3	3,7
	6	+7	+1	+3	+5	+1	-5	+2	+5	+10	+2	+3	4,0
	5	-2	+1	+3	+3	-4	+6	-2	+4	-6	+4	+1	4,0
	4	-1	-3	-4	-4	+2	+0	+1	-7	+0	-5	-2	2,9
	3	+2	+2	+3	-1	+3	-4	+2	-1	+3	-2	+1	2,5
	2	+3	+3	+2	+2	-4	+7	-1	+6	+2	+8	+3	3,6
	1	-4	-3	-3	-1	+2	-6	-2	-2	-3	-5	-3	2,2
% vs													
Norm of the X Matrix	Original	-0,04%	0,05%	0,00%	0,01%	-0,01%	0,01%	0,00%	-0,02%	0,04%	0,00%	0,02%	0,02%
VaR	Original	-0,76%	1,42%	-0,23%	-0,94%	2,99%	0,74%	0,50%	0,63%	5,85%	1,14%	1,52%	1,70%
	Total Nom	10,00%	10,22%	10,05%	9,98%	10,37%	10,15%	10,12%	10,14%	10,66%	10,19%	10,19%	0,20%
CVaR	Original	-0,96%	1,29%	-0,48%	-0,89%	2,91%	1,20%	0,04%	1,95%	5,47%	0,89%	1,61%	1,57%
	Total Nom	10,27%	10,50%	10,32%	10,28%	10,67%	10,49%	10,37%	10,57%	10,94%	10,46%	10,49%	0,20%

Table 12: All together's disturbances of the Example 2.

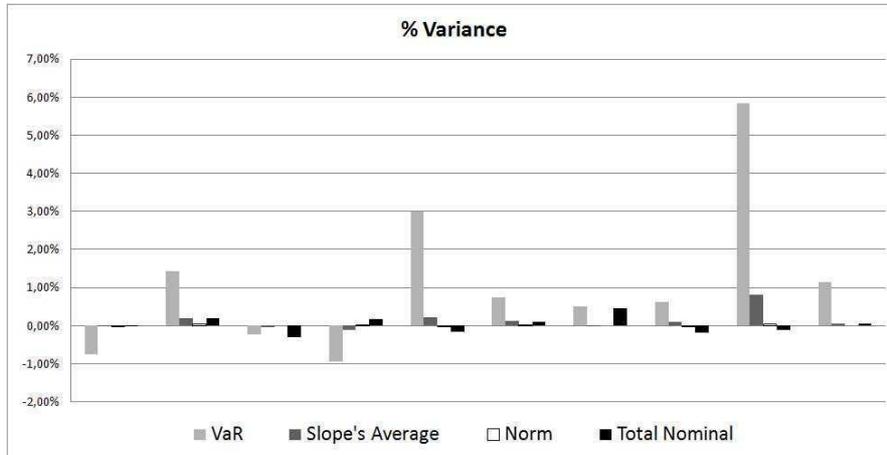


Figure 19: VaR vs Norm vs Nominal vs Survival's function of the Example 2.

6 Conclusions

Our main goal has been to alter the parameters of our portfolio in order to analyze them separately and together, and to determinate which of the variables are the most sensitive. As we have been explaining through the document, the four variables which have been modified do not behave the same way. We also would like to mention that all of our results are taken for a limited number of examples with a limited number of cases, so we have to be careful about our result's accuracy. The changes in the correlation matrix do not practically interfere in the final result. In fact, the standard deviations of VaR and CVaR due to this parameter are just 0.25% and 0.24%, and the averages are -0.04% and -0.15% . Finally, the VaR maximum percentages compared to the original ones that we have seen in this project are really low (from 0.35% to -0.52%).

This has not been the case of the risk disturbance or nominal. Risk' changes have a significant effect on VaR and CVaR: their respective averages and standard deviations are -0.70% and -0.57% , and 0.51% and 0.49%. As we have mentioned, we can consider that the biggest influence on the amount of losses, and consequently the sign of the averages, is due to higher risks. Talking about nominal, the risk measure averages have not been really high (0.06% and 0.16%, VaR and CVaR respectively) but the standard deviations have (0.47% and 0.40%).

We have not only appreciated relationships between these two parameters with the risk measures, but also between them. If the nominal invested in the portfolio is modified, we have observed that for equal quantities we obtain

different results. On the other hand, for similar changes in the client's risks we have observed enormous differences. The dependence on the amount of money related to each firm's risk is a really important matter in this project.

However, the most significant data is the disturbance in the survival function: while the average of VaR and CVaR disturbances are 0.80% and 0.94% respectively, the standard deviation are 3.12% and 3.20%. This means that they could affect the total amount of losses even six times more than risk disturbance could. In fact, as we have appreciated in the figures above, the VaR and CVaR's sign is caused by the increased or decreased of the high risks' slope function. Again, we should stress the importance of the higher risks as the risk measures are more sensitive to their changes in the survival probability function.

In order to value the model's stability, considering as the method used for each variable, which has been explained respectively in each section and disturbing a maximum of a 10% from the original data, we could accept any result except for variations approximately over 10%.

If we observe in the table 13 a summary of the data obtained in each case, we can appreciate that for disturbances under 10%, which means under the defined limit, the results obtained for the modifications of VaR and CVaR are strongly under the 10%. In fact, not even the maximums obtained exceed this percentage.

	NOMINAL		RISK		CORRELATION MATRIX		SURVIVAL FUNCTION		ALL TOGETHER EXAMPLE 1		ALL TOGETHER EXAMPLE 2	
	VaR	CVaR	VaR	CVaR	VaR	CVaR	VaR	CVaR	VaR	CVaR	VaR	CVaR
MIN	0,05%	0,02%	0,24%	0,12%	0,01%	0,01%	0,86%	0,62%	0,10%	0,20%	0,23%	0,04%
MAX	0,84%	0,68%	1,65%	1,28%	0,52%	0,78%	4,79%	4,85%	2,91%	3,19%	5,85%	5,47%
AVERAGE	0,40%	0,37%	0,68%	0,62%	0,19%	0,17%	2,86%	2,93%	1,69%	1,87%	1,52%	1,61%

Table 13: Minimum, maximum and average of the data obtained in each case.

It is worth noting that the disturbances related to clients' nominal, clients' risk and the norm of the correlation matrix hardly represent modifications in VaR and CVaR. On the other hand, the clearest case of disturbance in VaR and CVaR occurred when the survival function's slope is modified. In this case, it is possible to reach mean values of 2,86% and 2,93% for VaR and CVaR respectively. The biggest change is obtained modifying every parameter consider in the portfolio at the same time in the second example, reaching maximum values for both VaR and CVaR of 5,85% and 5,47% and even then, these results do not exceed the 10% defined.

Therefore, with all the evidences described before, we can affirm that our model is stable.

Acknowledgments

This work has been done in the framework of the project MTM2008-04621/MTM of the Spanish "Ministry of Science and Innovation" (National Plan of I+D+i 2008-2011); the project CONS-C6-0356 of the "I-MATH Proyecto Ingenio Mathematica"; the "Comunidad de Madrid" through the QUIMAPRES project S2009/PPQ-1551; and the Consolidation Project of Research Groups funded by the "Banco Santander" and the "Universidad Complutense de Madrid" (Ref. 910480).

Appendix: Glossary

We list here the main definitions that we use in this article [9, 10]:

- **Asset:** A resource with economic value that an individual, corporation or country owns or controls with the expectation that it will provide future benefit.
- **Asset-Backed Security (ABS):** A financial security backed by a loan, lease or receivables against assets other than real estate and mortgage-backed securities. For investors, asset-backed securities are an alternative to investing in corporate debt.
- **Bond:** A debt investment in which an investor loans money to an entity (corporate or governmental) that borrows the funds for a defined period of time at a fixed interest rate. Bonds are used by companies, municipalities, states and governments to finance a variety of projects and activities.
- **Capital:** Financial assets or the financial value of assets, such as cash.
- **Counterparty:** The other party that participates in a financial transaction. Every transaction must have a counterparty in order for the transaction to go through. More specifically, every buyer of an asset must be paired up with a seller that is willing to sell and vice versa.
- **Credit:** A contractual agreement in which a borrower receives something of value now and agrees to repay the lender at some date in the future, generally with interest. The term also refers to the borrowing capacity of an individual or company.
- **Default:** The failure to promptly pay interest or principal when due. Default occurs when a debtor is unable to meet the legal obligation of

debt repayment. Borrowers may default when they are unable to make the required payment or are unwilling to honor the debt.

- **Equity:** A stock or any other security representing an ownership interest.
- **Facility:** A term used to describe financial assistance programs offered by lending institutions to help companies requiring capital.
- **Liability:** A company's legal debts or obligations that arise during the course of business operations. Liabilities are settled over time through the transfer of economic benefits including money, goods or services.
- **Liquidity:** The degree to which an asset or security can be bought or sold in the market without affecting the asset's price. Liquidity is characterized by a high level of trading activity. Assets that can be easily bought or sold, are known as liquid assets.
- **Loan:** The act of giving money, property or other material goods to another party in exchange for future repayment of the principal amount along with interest or other finance charges.
- **Loss:** The difference between the revenue received from the sale of an output and the opportunity cost of the inputs used.
- **Maturity:** The length of time until the principal amount of a bond must be repaid.
- **Nominal value:** The stated value of an issued security that remains fixed, as opposed to its market value, which fluctuates.
- **Obligation:** The legal responsibility to meet the terms of a contract. If the obligation is not met there is often recourse for the other party to the contract.
- **Payoff:** The act or occasion of receiving money or material gain especially as compensation or as a bribe.
- **Portfolio:** A grouping of financial assets such as stocks, bonds and cash equivalents, as well as their mutual, exchange-traded and closed-fund counterparts. Portfolios are held directly by investors and/or managed by financial professionals.

- **Rating:** An evaluation of a corporate or municipal bond's relative safety from an investment standpoint. Basically, it scrutinizes the issuer's ability to repay principal and make interest payments. Bonds are rated by various organizations such as S&P and Moody's. Ratings range from AAA or Aaa (the highest), to C or D, which represents a company that has already defaulted.
- **Recovery rate:** The amount that a creditor would receive in final satisfaction of the claims on a defaulted credit.
- **Return:** The gain or loss of a security in a particular period. The return consists of the income and the capital gains relative on an investment.
- **Risk:** The chance that an investment's actual return will be different than expected. This includes the possibility of losing some or all of the original investment. Risk is usually measured by calculating the standard deviation of the historical returns or average returns of a specific investment. A fundamental idea in finance is the relationship between risk and return. The greater the amount of risk that an investor is willing to take on, the greater the potential return. The reason for this is that investors need to be compensated for taking on additional risk.
- **Security:** An instrument representing ownership (stocks), a debt agreement (bonds) or the rights to ownership (derivatives).
- **Security interest:** A legal claim or collateral that has been pledged, usually to obtain a loan. The borrower provides the lender with a security interest on certain securities/assets which can be repossessed in the event that timely obligation payments are not met.
- **Spread:** The difference between the bid and the ask price of a security or asset. The spread for an asset is influenced by a number of factors (the total number of shares outstanding that are available to trade, demand or interest in a stock, total trading activity of the stock...).
- **Tranches:** A piece, portion or slice of a deal or structured financing. This portion is one of several related securities that are offered at the same time but have different risks, rewards and/or maturities. "Tranche" is the French word for "slice".
- **Volatility:** A statistical measure of the dispersion of returns for a given security or market index. Volatility can either be measured by

using the standard deviation or variance between returns from that same security or market index.

References

- [1] David X. Li, *On Default Correlation: A Copula Function Approach*, Journal of Fixed Income, Vol. 9, No. 4, pp. 43-54, 2000.
- [2] Duan, J. C., *Maximum Likelihood Estimation Using Price Data of the Derivative Contract*, Mathematical Finance 4, pages 155-157, 1994.
- [3] Elizalde Abel, *Credit Risk Models I: Default Correlation in Intensity Models*, CEMFI and UPNA, 2005.
- [4] Elizalde Abel, *Credit Risk Models II: Structural Models*, CEMFI and UPNA, 2005.
- [5] Black, F., and J. C. Cox, *Valuing Corporate Securities: Some Effects of Bond Indenture Provisions*, Journal of Finance 31, pages 351-367, 1976.
- [6] Black, F., and Scholes, M., *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy 81, pages 637-654, 1973.
- [7] Eom, Y. H., Helwege, J., and Huang, J. Z., *Structural Models of Corporate Bond Pricing: An Empirical Analysis*, Review of Financial Studies 17, pages 499-544, 2003.
- [8] H. Fischer, *The Central Limit Theorem from Laplace to Cauchy*, 2000.
- [9] www.investopedia.com.
- [10] www.investorwords.com.
- [11] Giesecke, K., and Goldberg, L. R., *Sequential defaults and incomplete information*, Journal of risk 7, pages 1-26, 2004.
- [12] Ivorra Benjamin, *Optimisation globale semi-déterministe et applications industrielles*, Ph.D. Thesis, Université de Montpellier 2, 2006.

- [13] Ivorra Benjamin, Bijan Mohammadi, Angel Manuel Ramos, *Semideterministic Global Optimization Method: Application to a Control Problem of the Burgers Equation*, Journal of Optimization, Theory and Applications (J. Optimiz. Theory App.), ISSN: 0022-3239. Vol. 135, 2007, pages 549-561, 2007.
- [14] Jones, P., Mason, S., and Rosenfeld, E., *Contingent Claim Analysis of Corporate Capital Structures: An Empirical Investigation*, Journal of Finance 39, pages 611-625, 1984.
- [15] Merton, R., *On the Pricing of Corporate Debt: the Risk Structure of Interest Rates*, Journal of Finance 29, pages 449-470, 1974.
- [16] M. Sellers and A. Davidson, *Modelling default risk: Private firm model*, KMV Corporation, 1998.
- [17] P.J. Schnbucher, *Credit Derivatives Pricing Models*, Wiley Finance, 2003.
- [18] Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, David Heath, *Coherent Measures of Risk*, Blackwell Publishers Inc, Volume 9 Issue 3, pages 203-228, 1999.
- [19] R. Tyrrell Rockafellar, Stanislav Uryasev, *Conditional value-at-risk for general loss distributions*, Journal of banking and finance, ISSN 0378-4266, Vol. 26, N. 7, pages 1443-1471, 2002.
- [20] R. Cavestany Sanz-Briz, *Estructuras de Correlaciones de Crédito para el Cálculo de Capital Económico*, PriceWaterHouse Coopers, 2002.

**PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA**
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2009

1. DESIGN OF CODE DIVISION MULTIPLE ACCESS FILTERS USING GLOBAL OPTIMIZATION TECHNIQUES, B. Ivorra, B. Mohammadi, and A. M.Ramos
2. DYNAMICS IN DUMBBELL DOMAINS II. THE LIMITING PROBLEM, J.M. Arrieta, A. N. Carvalho and G. Lozada-Cruz
3. DYNAMICS IN DUMBBELL DOMAINS III. CONTINUITY OF ATTRACTORS , J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz
4. GEOMETRIC VERSUS SPECTRAL CONVERGENCE FOR THE NEUMANN LAPLACIAN UNDER EXTERIOR PERTURBATIONS OF THE DOMAIN , J. M. Arrieta and D. Krejcirik
5. ON THE MODELLING AND SIMULATION OF HIGH PRESSURE PROCESSES AND INACTIVATION OF ENZYMES IN FOOD ENGINEERING, J.A. Infante, B. Ivorra, Á.M. Ramos and J.M. Rey
6. CHARACTERIZATION OF THE MARTIAN SURFACE LAYER, G.Martínez, F. Valero, L. Vazquez.
7. CHARACTERIZATION OF THE MARTIAN CONVECTIVE BOUNDARY LAYER, G.Martínez, F. Valero, L. Vazquez.
8. INFINITE RESONANT SOLUTIONS AND TURNING POINTS IN A PROBLEM WITH UNBOUNDED BIFURCATION, J.M. Arrieta, R.Pardo, A.Rodríguez—Bernal
9. CASCADES OF HOPF BIFURCATIONS FROM BOUNDARY DELAY, J.M. Arrieta, N. Cónsul, S. Oliva
10. QUENCHING PHENOMENA FOR A NON-LOCAL DIFFUSION EQUATION WITH A SINGULAR ABSORPTION, R. Ferreira
11. WELL-POSEDNESS OF THE EINSTEIN–EULER SYSTEM IN ASYMPTOTICALLY FLAT SPACETIMES: THE CONSTRAINT EQUATIONS, U. Brauer and L. Karp
12. WELL-POSEDNESS OF THE EINSTEIN–EULER SYSTEM IN ASYMPTOTICALLY FLAT SPACETIMES: THE EVOLUTION EQUATIONS, U. Brauer and L. Karp
13. EXTREMAL EQUILIBRIA FOR MONOTONE SEMIGROUPS IN ORDERED SPACES WITH APPLICATIONS TO EVOLUTIONARY EQUATIONS, J. W. Cholewa and A. Rodriguez-Bernal
14. ASYMPTOTIC BEHAVIOUR OF A PARABOLIC PROBLEM WITH TERMS CONCENTRATED IN THE BOUNDARY, A. Jiménez-Casas and A. Rodriguez-Bernal
15. ASYMPTOTIC BEHAVIOR OF REACTION DIFFUSION EQUATIONS IN WEIGHTED SOBOLEV SPACES, J.M. Arrieta, N. Moya and A. Rodriguez-Bernal

**PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2010**

1. CONTINUITY OF DYNAMICAL STRUCTURES FOR NON-AUTONOMOUS EVOLUTION EQUATIONS UNDER SINGULAR PERTURBATIONS, J. Arrieta, A. N. Carvalho, J. Langa and A. Rodríguez-Bernal.
2. ON THE LONG TIME BEHAVIOUR OF NON-AUTONOMOUS LOTKA-VOLTERRA MODELS WITH DIFFUSION VIA THE SUB-SUPER TRAJECTORY METHOD, J.A. Langa, A. Rodríguez-Bernal and A. Suárez.
3. MODELLING AND SIMULATION OF A POLLUTED WATER PUMPING PROCESS, C. Alavani, R. Glowinski, S. Gomez, B.Ivorra, P. Joshi and A. M. Ramos.
4. EXPANDING THE ASYMPTOTIC EXPLOSIVE BOUNDARY BEHAVIOR OF LARGE SOLUTIONS TO A SEMILINEAR ELLIPTIC EQUATION, S. Alarcón, G. Díaz, R. Letelier and J. M. Rey.
5. A SINGULAR PERTURBATION IN A LINEAR PARABOLIC EQUATION WITH TERMS CONCENTRATING ON THE BOUNDARY, A. Rodríguez-Bernal.
6. ISOTHERMALISATION FOR A NON-LOCAL HEAT EQUATION, E. Chasseigne and R. Ferreira.
7. HOMOGENIZATION IN A THIN DOMAIN WITH AN OSCILLATORY BOUNDARY, J. M. Arrieta and M. C. Pereira
8. VERY RAPIDLY VARYING BOUNDARIES IN EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS. THE CASE OF A NON UNIFORMLY LIPSCHITZ DEFORMATION, J.M. Arrieta and S. Bruschi
9. PERTURBATION OF ANALYTIC SEMIGROUPS IN SCALES OF BANACH SPACES AND APPLICATIONS TO PARABOLIC EQUATIONS WITH LOW REGULARITY DATA, A. Rodríguez-Bernal
10. IDENTIFICATION OF A PRESSURE DEPENDENT HEAT TRANSFER COEFFICIENT, A. Fraguera, J. A. Infante, Á. M. Ramos and J. M. Rey.
11. MATHEMATICAL MODELING FOR PROTEIN FOLDING DEVICES. APPLICATIONS TO HIGH PRESSURE PROCESSING AND MICROFLUIDIC MIXERS, J. Bello Rivas, J. A. Infante, B. Ivorra, J. López Redondo, P. Martínez Ortigosa, A. M. Ramos, J. M. Rey, and N. Smith
12. A VARIANCE-EXPECTED COMPLIANCE APPROACH FOR TOPOLOGY OPTIMIZATION, M. Carrasco, B. Ivorra, R. Lecaros and A. M. Ramos
13. MODELING AND SIMULATION OF CLASSICAL SWINE FEVER VIRUS SPREAD BETWEEN AND WITHIN FARMS, B.Ivorra, B.Martinez-Lopez, J. M. Sanchez-Vizcaino and A.M. Ramos
14. NONLINEAR PARABOLIC PROBLEMS IN THIN DOMAINS WITH A HIGHLY OSCILLATORY BOUNDARY, J. Arrieta, A. C. Carvalho, M. C. Pereira and R. P. Silva.
15. SINGULAR LIMIT FOR A NONLINEAR PARABOLIC EQUATION WITH TERMS CONCENTRATING ON THE BOUNDARY, A. Jiménez-Casas and A. Rodríguez-Bernal

16. SENSITIVITY ANALYSIS OF A DEFAULT TIME MODEL FOR CREDIT RISK PORTFOLIO MANAGEMENT, R. Abella Muñoz, I. Armero Huertas, B. Ivorra and A. M. Ramos del Olmo