DYNAMICS IN DUMBBELL DOMAINS II. THE LIMITING PROBLEM

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Abstract. In this work we continue the analysis of the asymptotic dynamics of reaction diffusion problems in a dumbbell domains started in [3]. Here we study the limiting problem, that is, an evolution problem in a "domain" which consists of an open, bounded and smooth set $\Omega \subset \mathbb{R}^N$ with a curve R_0 attached to it. The evolution in both parts of the domain is governed by a parabolic equation. In Ω the evolution is independent of the evolution in R_0 whereas in R_0 the evolution depends of the evolution in Ω through the continuity condition of the solution at the junction points. We analyze in detail the linear elliptic and parabolic problem, the generation of linear and nonlinear semigroups, the existence and structure of attractors.

1. Introduction

In this paper we continue the analysis of the asymptotic dynamics of parabolic equations in dumbbell type domains initiated in [3]. More precisely, in [3] we started the analysis of a parabolic equation of the form

$$\begin{cases} u_t - \Delta u + u = f(u), \quad x \in \Omega_\epsilon,\ t > 0, \\ \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega_\epsilon, \end{cases} \tag{1.1}$$

where $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geqslant 2$, is a typical dumbbell domain consisting of two disconnected domains, that we will denote by Ω, joined by a thin channel, R_ϵ, which degenerates to a line segment as the parameter ϵ approaches zero, see Figure 1.

The limit "domain" will consist of the open set Ω and the line segment R_0, that without loss of generality we may assume that $R_0 = \{(x, 0, \ldots, 0) : 0 < x < 1\}$, see Figure 2.

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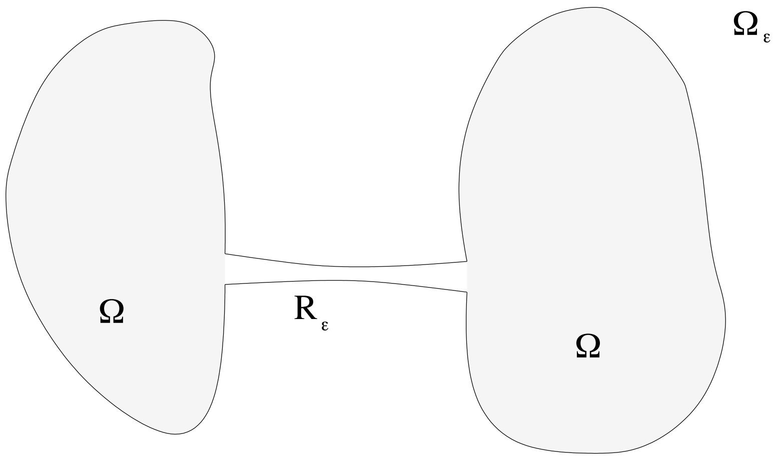

FIGURE 1. Dumbbell domain

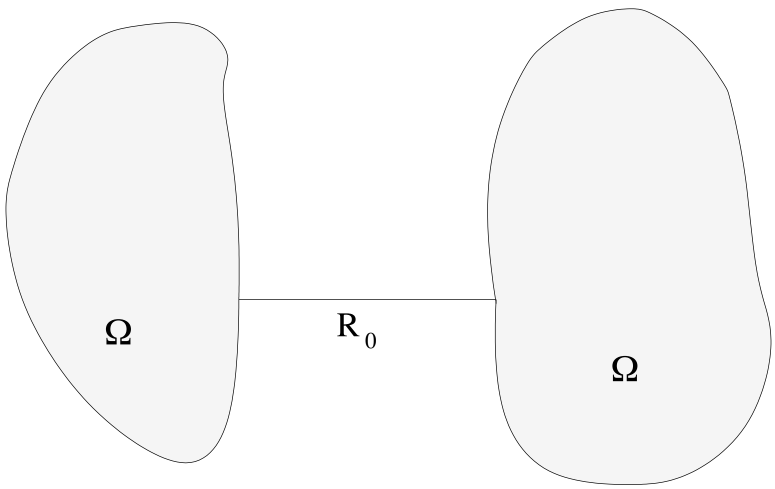

FIGURE 2. Limit "domain"

The limit equation is given by

$$\begin{cases} w_t - \Delta w + w = f(w), \quad x \in \Omega,\ t > 0 \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \\ \\ v_t - \dfrac{1}{g}(g v_x)_x + v = f(v), \quad x \in (0,1) \\ v(0) = w(P_0),\ v(1) = w(P_1) \end{cases} \tag{1.2}$$

where w is a function that is defined in Ω, v is defined in the line segment R_0 and the points $P_0 = (0, 0, \ldots, 0)$, $P_1 = (1, 0, \ldots, 0)$ are the points of junction of the line segment with the open set Ω, see [3]. Observe that the boundary conditions of the function v are given in terms of a continuity condition at P_0 and P_1, so that (w, v) seen as a function defined on $\Omega \cup \overline{R_0}$ is continuous.

The function g is related to the geometry of the channel R_ϵ, more exactly, on the way the channel R_ϵ collapses to the line segment R_0. This is the case, for instance, in two dimensions, if the channel $R_\epsilon = \{(x, y) : 0 < x < 1, 0 < y < \epsilon g(x)\}$, although more general and complicated geometries are allowed, see [3].

Let us briefly describe the appropriate functional analytic framework that we developed in [3] to treat this singular perturbation problem. For $0 < \epsilon \leqslant 1$, let $U^p_\epsilon := L^p(\Omega_\epsilon)$, with the norm

$$\|u_\epsilon\|^p_{U^p_\epsilon} = \int_\Omega |u|^p + \frac{1}{\epsilon^{N-1}} \int_{R_\epsilon} |u_\epsilon|^p$$

For $\epsilon = 0$, let $U^p_0 := L^p(\Omega) \oplus L^p_g(0, 1)$, that is $(w, v) \in U^p_0$ if $w \in L^p(\Omega)$, $v \in L^p(0, 1)$ and the norm is given by

$$\|(w, v)\|^p_{U^p_0} = \int_\Omega |w|^p + \int_0^1 g|v|^p.$$

We studied in [3] the convergence of the set of equilibria in these spaces. We note that the spaces change with the parameter and the notion of convergence must be carefully explained (see [3]). As a matter of fact we constructed the linear operators $A_\epsilon : D(A_\epsilon) \subset U^p_\epsilon \to U^p_\epsilon$ given by $A_\epsilon(u) = -\Delta u + u$ for $0 < \epsilon \leqslant 1$ where $D(A_\epsilon) = \{u \in W^{2,p}(\Omega_\epsilon) : \Delta u \in U^p_\epsilon,\ \partial u/\partial n = 0 \text{ in } \partial\Omega_\epsilon\}$, and $A_0 : D(A_0) \subset U^p_0 \to U^p_0$ given by $A_0(w, v) = (-\Delta u + u, -\frac{1}{g}(gv_x)_x + v)$ where $D(A_0) = \big\{(w, v) \in U^p_0 : w \in D(\Delta^\Omega_N),\ (gv_x)_x \in L^p_g(0, 1), v(0) = w(P_0),\ v(1) = w(P_1)\big\}$ and studied the convergence properties of A^{-1}_ϵ to A^{-1}_0, see Proposition 2.7 of [3]. Moreover, if

$$\begin{aligned}
&F_\epsilon(u_\epsilon)(x) = f(u_\epsilon(x)),\ x \in \Omega_\epsilon \\
&F_0(w, v) = (\bar w, \bar v),\ \text{ where } \begin{cases} \bar w(x) = f(w(x)),\ x \in \Omega \\ \bar v(x) = f(v(x)),\ x \in R_0, \end{cases}
\end{aligned}$$

considering the equilibria of (1.1) and (1.2) as fixed points of the nonlinear maps $A^{-1}_\epsilon \circ F_\epsilon : U^p_\epsilon \to U^p_\epsilon$ and of $A^{-1}_0 \circ F_0 : U^p_0 \to U^p_0$ respectively, for the appropriate nonlinearities, we showed the convergence of the equilibria see Theorem 2.3 of [3]. Also, in case the equilibrium of the limit problem (1.2) is hyperbolic, we proved the convergence of the linearizations around the equilibria and the convergence of the linear unstable manifolds.

As we mentioned in the introduction of [3], our final objective is to compare the whole dynamics of problems (1.1) and (1.2), that is, to compare the attractors of both problems and it is very clear from [3], that the spaces U^p_ϵ, U^p_0 provide a natural and appropriate functional framework to study and compare the dynamics of this perturbation problem.

In this paper we concentrate in analyzing the dynamics of the limit problem (1.2) in U^p_0. In fact, we will consider a problem which is more general, allowing nonlinearities depending on the spacial variable x and the points P_0, P_1 be arbitrary points in $\bar\Omega$, that is,

$$\begin{cases} w_t - \Delta w + w = f(x,w) & x \in \Omega,\ t>0 \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ v_t - \dfrac{1}{g}\left(g v_x\right)_x + v = f(x,v), & x \in (0,1) \\ v(0) = w(P_0), \quad v(1) = w(P_1) \end{cases} \tag{1.3}$$

where $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a C^1, dissipative nonlinearity. That is,

$$\sup_{x \in \Omega \cup R_0} \limsup_{|s|\to\infty} \frac{f(x,s)}{s} < 1. \tag{1.4}$$

We also assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : R_0 \times \mathbb{R} \to \mathbb{R}$ are continuous. This setting allows us to consider different nonlinearities in Ω and in R_0, say $f(x,s) = f_1(s)$ in Ω and $f(x,s) = f_2(s)$ in R_0.

Besides the fact that this problem appears in a natural way as the limit problem of a reaction diffusion equation in a dumbbell domain, it actually has some special features that make the study of its dynamics very interesting by itself. Let us briefly mention some of this interesting features:

(1) From the equations, it is clear that the variable w does not depend on the variable v. This means that all the interesting features of a usual parabolic problem with Neumann boundary conditions are present in (1.3). To observe them it is enough to ignore the variable v.

(2) On the other hand, the variable v depends on the behavior of w (one-sided coupling). The dependence of v upon w is obtained through the coupling at the boundary which requires that the function w has a well defined trace in a point. This makes things like generation of semigroups and local well posedness a more delicate matter.

(3) In the case $f(x,u) = f(u)$, consider the functions w and v spatially constant. The pair (w,v) will be a solution of (1.3) if and only if both are solutions of the ordinary differential equation $\dot{u} = -u + f(u)$. This says that the dissipativeness assumption should be indeed (1.4).

(4) Let u^* be a stable equilibria for the equation in Ω; that is, for the problem

$$\begin{cases} w_t - \Delta w + w = f(w) & x \in \Omega,\ t>0 \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

It is not automatic that we will have a stable equilibria for (1.3) which is of the form (u^*, v^*). A question that we address here is to find out when this is possible.

(5) If $\Omega = \Omega_1 \cup \Omega_2$ with $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$ and c_1, c_2 are fixed points of f which satisfy $f'(c_i) < 1$, $i = 1,2$, then $w^* = c_1\chi_{\Omega_1} + c_2\chi_{\Omega_2}$ is a stable nonconstant equilibria for (1.5). Is it possible to find conditions on f such that there is an stable equilibria for (1.2) of the form (w^*, v^*). We will see that this is the case and we will use it to give alternative proofs of existence of patterns.

One of the difficult points for the treatment of problem (1.2) in the space U_0^p is that, even though the operator A_0 generates a semigroup $T(t)$ in U_0^p with the property that $t \to T(t)(w,v)$ is continuous at $t=0$ for smooth data (w,v), it will not be continuous at $t=0$ for general $(w,v) \in U_0^p$. Actually, we show that

$$\|T(t)\|_{\mathcal{L}(U_0^p,U_0^p)} \leqslant Ct^{-1+\alpha}$$

for some $0<\alpha<1$ depending on p and N and that the singularity of this estimate at $t=0$ cannot be avoided. We will actually provide an example of an initial condition $(w_0,v_0) \in U_0^p$ satisfying that $\|T(t)(w_0,v_0)\|_{U_0^p} \geqslant Ct^{-\delta}$ as $t \to 0$, for some $\delta > 0$. This singular behavior at $t=0$ for general initial data is a consequence of a deficiency in the resolvent estimates associated to the operator A_0. Actually we have that the operator A_0 is a closed, densely defined operator satisfying the estimate

$$\|(A_0+\lambda I)^{-1}\|_{\mathcal{L}(U_0^p,U_0^p)} \leqslant \frac{C}{|\lambda|^\alpha + 1},$$

with λ in an appropriate sector in the complex plane and $\alpha \in (0,1)$. We say that this estimate is "deficient" since $\alpha < 1$. If it were $\alpha = 1$ we could apply the standard theory of generation of strongly continuous analytic semigroups. In turns, this deficiency comes from the continuity condition we need to impose on the function in $D(A_0)$ at the junction of Ω and R_0.

The singular behavior of the linear semigroup as $t \to 0$ prevents us from applying the standard theory on local existence of solutions for semilinear equations of the type $\dot{x}+A_0x = F(x)$, as developed for instance in the book [11] and we will need to draw some techniques and ideas developed in [2] to get solutions for the equations.

In Section 2 we prove some abstract results on generation of linear semigroups for operators having a deficiency in the resolvent estimate and give a local existence result for semilinear evolution equations with the linear operator presents the mentioned deficiency.

In Section 3 we apply the theory developed in the previous Section and study the generation of linear semigroup by the operator A_0. We analyze the singularity at $t=0$ of the semigroup and provide an example of an initial condition $(w_0,v_0) \in U_0^p$ so that $\|T(t)(w_0,v_0)\|_{U_0^p} \geqslant Ct^{-\delta}$ for some δ. We also study the spectrum of the operator A_0. We see that A_0 has compact resolvent, its eigenvalues are all real and nonnegative but A_0 does not have a selfadjoint structure. As a matter of fact we will see that it is possible, for some eigenvalues of A_0, that the algebraic and geometric multiplicity do not coincide. Recall that the spectra of A_0 is the limit of the spectra of the Laplace operator with Neumann boundary condition in the dumbbell domain Ω_ϵ as $\epsilon \to 0$. This has been shown in several works in the literature and in different situations. See for instance [12, 1, 9, 3] and references therein.

In Section 4 we analyze the the nonlinear problem (1.3). Once the linear operator and the properties of the linear semigroup are well understood, we are able to give a local and global existence results for nonlinear problems. We also study the regularization properties of the semigroup and the existence of the global attractor. We will also pay special attention to the structure of the attractor. We will not be able to construct a Lyapunov function but we will be able to show a gradient-like structure in the case where the system has a finite number of equilibria. In this case, as a consequence of the results in [6], the attractor is characterized as the unstable manifolds of equilibria. This structure will be particularly important when dealing with the continuity of the attractors in [4].

Finally, in Section 5 we consider several important comments on the asymptotic dynamics of equation (1.3).

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2. Abstract Theory of Semilinear Singular Semigroups

Let X be a Banach space and $A : D(A) \subset X \to X$ a closed densely defined operator. In this section we consider the evolutionary problem

$$\begin{cases} \dot{x} + Ax = f(x), \\ x(0) = x_0 \in X, \end{cases} \tag{2.1}$$

where the operator A has some deficiency in the resolvent estimate (as mentioned in the introduction) and f is an appropriate nonlinearity. We will see how this deficiency implies a singular behavior at $t = 0$ of the semigroup generated by A and in particular the semigroup is not a strongly continuous semigroup.

2.1. **The linear problem.** Let $A : D(A) \subset X \to X$ be a closed, densely defined operator. Assume that, for some $\theta \in (0, \frac{\pi}{2})$, we have $\rho(-A) \supset \Sigma_\theta$ where

$$\Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : \ |\arg \lambda| \leqslant \pi - \theta\} \cup \{0\} \tag{2.2}$$

and that, for some $0 < \alpha < 1$ we have the estimate,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C}{|\lambda|^\alpha + 1}, \ \forall \lambda \in \Sigma_\theta. \tag{2.3}$$

Consequently

$$\|A(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leqslant 1 + C|\lambda|^{1-\alpha}, \ \forall \lambda \in \Sigma_\theta. \tag{2.4}$$

Observe that from these estimates it is not possible to apply to A the general results and techniques on generation of strongly continuous semigroups, as it is developed in [11] or [14], for instance. Nonetheless if we let Γ be the boundary of the sector of Σ_θ, oriented in such a way that the imaginary part is increasing and if we define

$$T(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \ (\lambda + A)^{-1} \, d\lambda,$$

then, $T(t)$ will be our candidate for the semigroup generated by A. We start with some preliminary properties of $T(t)$.

Proposition 2.1. *We have,*
i) $\{T(t) : t > 0\} \subset \mathcal{L}(X)$ *and there is a constant* $C > 0$ *such that*

$$\|T(t)\|_{\mathcal{L}(X)} \leqslant Ct^{-1+\alpha}. \tag{2.5}$$

ii) $\{AT(t) : t > 0\} \subset \mathcal{L}(X)$, $AT(t)$ *is given by*

$$AT(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \, A \, (\lambda + A)^{-1} \, d\lambda.$$

and the following holds

$$\|AT(t)\|_{\mathcal{L}(X)} \leqslant C\max\{t^{-1},t^{-2+\alpha}\}. \tag{2.6}$$

iii) $\{T(t)\,:\,t>0\}$ *satisfies the semigroup property, that is* $T(t+s)=T(t)T(s)$ *for all* $t,s>0$.

Proof: *i)* Observe first that the integral

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,(\lambda+A)^{-1}\,d\lambda,$$

converges in the uniform operator topology of $\mathcal{L}(X)$ for all $t>0$. In fact,

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,(\lambda+A)^{-1}\,d\lambda\right\|_{\mathcal{L}(X)} &\leqslant \frac{1}{2\pi}\left|\int_\Gamma e^{-\cos\theta|\lambda|t}\,\left\|(\lambda+A)^{-1}\right\|_{\mathcal{L}(X)}d|\lambda|\right|\\ &\leqslant \frac{1}{2\pi}\left|\int_\Gamma e^{-\cos\theta|\lambda|t}\frac{C}{|\lambda|^\alpha+1}d|\lambda|\right| \leqslant \frac{t^{\alpha-1}}{2\pi}\left|\int_\Gamma e^{-\cos\theta|\mu|}\frac{C}{|\mu|^\alpha}d|\mu|\right| \leqslant Ct^{\alpha-1},\end{aligned}$$

where we have performed the change of variables $\mu=\lambda t$.

ii) Since the operator A is closed and since, from (2.4), the integral

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,A\,(\lambda+A)^{-1}\,d\lambda,$$

is convergent we have that $AT(t)=\dfrac{1}{2\pi i}\displaystyle\int_\Gamma e^{\lambda t}\,A\,(\lambda+A)^{-1}\,d\lambda\in\mathcal{L}(X)$ for all $t>0$ and

$$\begin{aligned}\|AT(t)\|_{\mathcal{L}(X)} = \left\|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\,A\,(\lambda+A)^{-1}\,d\lambda\right\|_{\mathcal{L}(X)} &\leqslant \frac{1}{2\pi}\left|\int_\Gamma e^{-\cos\theta|\lambda|t}\,\left\|A\,(\lambda+A)^{-1}\right\|_{\mathcal{L}(X)}d|\lambda|\right|\\ &\leqslant \frac{1}{2\pi}\left|\int_\Gamma e^{-\cos\theta|\lambda|t}C(1+|\lambda|^{1-\alpha})d|\lambda|\right| = \frac{1}{2\pi}\left|\int_\Gamma e^{-\cos\theta|\mu|}C(1+t^{\alpha-1}|\mu|^{1-\alpha})t^{-1}d|\mu|\right|\\ &\leqslant \max\{t^{-1},t^{-2+\alpha}\}\frac{1}{2\pi}\left|\int_\Gamma e^{-\cos\theta|\mu|}C(1+|\mu|^{1-\alpha})d|\mu|\right| \leqslant C\max\{t^{-1},t^{-2+\alpha}\}.\end{aligned}$$

iii) The semigroup property, $T(t+s)=T(t)T(s)$ for all $t,s>0$ can be proved as in the case when $-A$ generates an usual analytic semigroup, see [11]. ■

Remark 2.2. *i) In the literature (see* [13] *and references there in) these semigroups have been called* Semigroup of Growth Order $(1-\alpha)$.

ii) Note that, for $0<t<1$ *we have*

$$\|AT(t)\|_{\mathcal{L}(X)} \leqslant Mt^{-2+\alpha}.$$

iii) If we assume that Y *is another Banach space,* $(\lambda+A)^{-1}\in\mathcal{L}(Y,X)$ *and that there exists* $\beta>0$ *such that*

$$\|(\lambda+A)^{-1}\|_{\mathcal{L}(Y,X)} \leqslant \frac{C}{|\lambda|^\beta+1},\ \forall\lambda\in\Sigma_\theta$$

we can prove in a similar way

$$\|T(t)\|_{\mathcal{L}(Y,X)} \leqslant Ct^{-1+\beta}.$$

Remark 2.3. *If we consider the following curve in the complex plane,*

$$\Gamma_R = \{z \in \mathbb{C}, |\arg(z)| = \pi - \theta, |z| \geqslant R\} \cup \{z \in \mathbb{C}, |\arg(z)| \leqslant \pi - \theta, |z| = R\} \subset \Sigma_\theta \tag{2.7}$$

oriented in such a way that the imaginary part is increasing, then $T(t)$ can also be expressed as

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma_R} e^{\lambda t}\ (\lambda + A)^{-1}\, d\lambda,$$

for any $R > 0$. The reason for this is that the region enclosed between Γ_R and $\Gamma(= \Gamma_0)$ does not contain any point of the spectra of A_0.

It is clear that $T(t)$ satisfies the semigroup properties but strong continuity fails at $t = 0$ for data which are not sufficiently smooth. Nonetheless, several of the properties of analytic semigroup will still hold for sufficiently regular data. We say that $\{T(t) : t \geqslant 0\}$ is the semigroup generated by A and do not make any allusion to continuity.

In what follows we derive some simple properties of the semigroup $\{T(t) : t \geqslant 0\}$ that we will employ to obtain a local well posedness result for the semilinear problem (1.3).

Our next lemma is saying that, in some sense, the semigroup $\{T(t) : t \geqslant 0\}$ is the solution operator for the linear differential problem

$$\begin{cases} \dot{x} + Ax = 0 \\ x(0) = x_0 \in X. \end{cases}$$

Lemma 2.4. *The semigroup $T(t) : (0, \infty) \to \mathcal{L}(X)$ is differentiable and*

$$\frac{d}{dt}T(t) = \frac{1}{2\pi i}\int_{\Gamma} \lambda e^{\lambda t}\ (\lambda + A)^{-1}\, d\lambda.$$

In addition, for each $x_0 \in X$, we have that

$$\frac{d}{dt}T(t)x_0 + AT(t)x_0 = 0,\ \ t > 0.$$

Proof: Since we are considering only $t > 0$, the proof is the usual one for analytic semigroups. ■

Next we prove that $T(t)x_0$ is continuous at $t = 0$ for each $x_0 \in D(A)$. To show this, we start by proving a technical result, which is known to hold for generators of strongly continuous semigroups but also holds for operators A which satisfy (2.3) for some $0 < \alpha < 1$.

Lemma 2.5. *If A is as before, then*

$$\|\lambda(\lambda + A)^{-1}A^{-1}\|_{\mathcal{L}(X)} \leqslant C,\ \forall \lambda \in \Sigma_\theta$$

and

$$\lambda(\lambda + A)^{-1}A^{-1}x \overset{|\lambda| \to \infty}{\longrightarrow} A^{-1}x.$$

Proof: In fact, for each $x \in X$ we have that

$$\|\lambda(\lambda + A)^{-1}A^{-1}x - A^{-1}x\|_X = \|(\lambda + A)^{-1}x\|_X \leqslant \frac{C}{|\lambda|^\alpha + 1}\|x\|_X \tag{2.8}$$

and the result follows. ■

We can show now,

Proposition 2.6. *If* $x \in D(A)$ *then* $\|T(t)x - x\|_X \to 0$ *as* $t \to 0$.

Proof: Observe that $x \in D(A)$ is equivalent to say that $x = A^{-1}y$ for some $y \in X$. Moreover, for any $R > 0$, we have

$$T(t)x - x = \frac{1}{2\pi i}\int_{\Gamma_R} e^{\lambda t}[(\lambda + A)^{-1} - \frac{1}{\lambda}I]A^{-1}yd\lambda = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{e^{\lambda t}}{\lambda}[\lambda(\lambda + A)^{-1}A^{-1}y - A^{-1}y]d\lambda$$

But, with (2.8) we have

$$\|T(t)x - x\|_X \leqslant \frac{1}{2\pi}\left|\int_{\Gamma_R} \frac{|e^{\lambda t}|}{|\lambda|}\frac{C}{|\lambda|^\alpha + 1}d|\lambda|\right| \cdot \|y\|$$

The curve Γ_R can be expressed as the union of two different parts $\Gamma_R = \Gamma_R^0 \cup \Gamma_R^1$ where $\Gamma_R^0 = \{z \in \mathbb{C}, |\arg(z)| = \pi - \theta, |z| \geqslant R\}$, $\Gamma_R^1 = \{z \in \mathbb{C}, |\arg(z)| \leqslant \pi - \theta, |z| = R\}$. We can see that, for fixed R, $|e^{\lambda t}| \leqslant 1$ in Γ_R^0 and $|e^{\lambda t}| \to 1$ as $t \to 0$ uniformly in $\lambda \in \Gamma_R^1$. Hence, for all $R > 0$, we have

$$\limsup_{t\to 0}\|T(t)x - x\|_X \leqslant \frac{C}{2\pi}\left|\int_{\Gamma_R}\frac{1}{|\lambda|(|\lambda|^\alpha + 1)}d|\lambda|\right| \cdot \|y\|$$

and we can easily see that

$$\left|\int_{\Gamma_R}\frac{1}{|\lambda|(|\lambda|^\alpha + 1)}d|\lambda|\right| \to 0, \text{ as } R \to +\infty$$

which proves the result. ∎

Our next result states that, in some sense, A is the generator of $T(t)$.

Proposition 2.7. *Assume that* $x \in D(A^2)$. *Then,*

$$\lim_{t\to 0^+}\frac{T(t)x - x}{t} + Ax = 0.$$

Proof: First note that $(\lambda + A)^{-1} - \frac{1}{\lambda}I = -\frac{1}{\lambda}(\lambda + A)^{-1}A$. If we consider the curve $\Gamma_{1/t}$ for $t > 0$ where Γ_R is defined in (2.7), then

$$\begin{aligned}\frac{T(t)x - x}{t} &= \frac{1}{2\pi i}\int_{\Gamma_{1/t}} e^{\lambda t}\frac{1}{t}[(\lambda + A)^{-1} - \frac{1}{\lambda}I]xd\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma_{1/t}} e^{\lambda t}\frac{1}{\lambda t}(\lambda + A)^{-1}Axd\lambda.\end{aligned}$$

With the change of variables $\mu = \lambda t$, which transforms $\Gamma_{1/t}$ into Γ_1, the above becomes

$$\begin{aligned}\frac{T(t)x - x}{t} &= -\frac{1}{2\pi i}\int_{\Gamma_1} e^{\mu}\frac{1}{\mu^2}\frac{\mu}{t}(\frac{\mu}{t} + A)^{-1}Axd\mu \\ &= -\frac{1}{2\pi i}\int_{\Gamma_1} e^{\mu}\frac{1}{\mu^2}\frac{\mu}{t}(\frac{\mu}{t} + A)^{-1}A^{-1}A^2xd\mu\end{aligned}$$

But from Lemma 2.5, we have that $\|\frac{\mu}{t}(\frac{\mu}{t}+A)^{-1}A^{-1}\| \leqslant C$ and therefore, the integrand in the above integral can be estimated by

$$\|e^{\mu}\frac{1}{\mu^2}\frac{\mu}{t}(\frac{\mu}{t}+A)^{-1}A^{-1}A^2x\| \leqslant C|e^{\mu}|\frac{1}{|\mu|^2}\|A^2x\|_X.$$

Moreover, again from Lemma 2.5, we have $\frac{\mu}{t}(\frac{\mu}{t}+A)^{-1}Ax \xrightarrow{t\to 0} Ax$. Since

$$\left|\int_{\Gamma_1} C|e^{\mu}|\frac{1}{|\mu|^2}\|A^2x\|_X\, d|\mu|\right| < \infty$$

and with the aid of the Dominated Convergence Theorem we get

$$\frac{1}{2\pi i}\int_{\Gamma_1} e^{\mu}\frac{1}{\mu^2}\frac{\mu}{t}(\frac{\mu}{t}+A)^{-1}Axd\mu \xrightarrow{t\to 0} -\frac{1}{2\pi i}\int_{\Gamma_1} e^{\mu}\frac{1}{\mu^2}Axd\mu = -\left(\frac{1}{2\pi i}\int_{\Gamma_1} e^{\mu}\frac{1}{\mu^2}d\mu\right)Ax$$

Using now residues theory and standard complex integration techniques we easily get that $\frac{1}{2\pi i}\int_{\Gamma_1} e^{\mu}\frac{1}{\mu^2}d\mu = 1$, which shows the result. ∎

2.2. **The semilinear problem.** In this subsection we consider the semilinear problem

$$\begin{cases} \dot{x}+Ax = f(x), \\ x(0) = x_0 \in X \end{cases} \tag{2.9}$$

where the operator A satisfies the deficiency in the resolvent as in the previous subsection. In particular, we have that (2.3) and (2.4) are satisfied.

We will also assume that we have another Banach space Y and that Remark 2.2 iii) holds, for some $\beta > 0$. Assume also that the nonlinearity $f : X \to Y$ is a locally Lipschitz and bounded map which satisfies,

$$\begin{aligned} &\|f(x)\|_Y \leqslant c(1+\|x\|_X^{\rho}), \\ &\|f(x)-f(y)\|_Y \leqslant c(1+\|x\|_X^{\rho-1}+\|y\|_X^{\rho-1})\|x-y\|_X, \end{aligned} \tag{2.10}$$

where $1 \leqslant \rho < \frac{\beta}{1-\alpha}$.

Remark 2.8. *In many instances, the relation between spaces X and Y is given by $X = Y^{\gamma}$ for some $0 \leqslant \gamma \leqslant 1$, that is, X is a fractional power spaces associated to the realization of the operator A in the Banach space Y.*

Our first task is to give meaning to a solution of the problem (2.9).

Definition 2.9. *We will say that $x(\cdot) : (0,\tau) \to X$ is a solution for the initial value problem (2.9) if $[0,\tau) \ni t \mapsto x(t) - T(t)x_0 \in X$ is continuous and*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))ds. \tag{2.11}$$

Remark 2.10. *Observe that we do not require the solution to be continuous in X at $t = 0$ and in general the solution will not be continuous at $t = 0$. This is the case, for instance, if $f \equiv 0$ so that we have that $x(t,x_0) = T(t)x_0$, which is not continuous at $t = 0$.*

We are able to show the following result, which is obtained very much in the spirit of the results in [2]

Proposition 2.11. *In the conditions above, for each $x_0 \in X$ there is a unique solution $x(\cdot, x_0) = T(\cdot, x_0)$ of (2.9) defined on a maximal interval of existence $(0, \tau_{\max}(x_0))$.*

Moreover, we have
i) The time of existence $\tau_{max}(x_0)$ can be chosen uniformly in bounded sets of X, in particular the following continuation result holds: either $\tau_{max}(x_0) = +\infty$ or $\limsup_{t\to\tau_{\max}} \|x(t, x_0)\|_X = +\infty$.
ii) The time of existence is upper semicontinuous in X, that is, if $x_n \to x_0$ in X then $\liminf_{n\to\infty} \tau_{max}(x_n) \geqslant \tau_{max}(x_0)$.
iii) The solution is continuous with respect to the initial conditions in the following sense: if $x_0 \in X$ and if $\tau < \tau_{max}(x_0)$, then for $\delta > 0$ small we have

$$\|x(t, x_0) - x(t, x_0')\|_X \leqslant Ct^{\alpha-1}\|x_0 - x_0'\|_X,\ t \in (0, \tau],\ \|x_0 - x_0'\|_X < \delta. \tag{2.12}$$

Proof: Since the linear part is singular at $t = 0$ we search for solutions for the semilinear problem with the same kind of singularity; that is, we seek for solutions in

$$K(\tau_0, x_0) = \{x \in C((0, \tau_0], X) : \sup_{t\in(0,\tau_0]} \|x(t) - T(t)x_0\|_X \leqslant \mu\},$$

with the metric

$$\|x - y\|_{K(\tau_0,x_0)} = \sup_{t\in(0,\tau_0]} \|x(t) - y(t)\|_X.$$

It is not difficult to see that, with this metric, $K(\tau_0, x_0)$ is a complete metric space.

Assume that $x_0 \in X$ and on $K(\tau_0, x_0)$ define the map

$$(U(x))(t) = T(t)x_0 + \int_0^t T(t - s)f(x(s))ds.$$

For suitably chosen $\tau_0 > 0$, we will show that, U takes $K(\tau_0, x_0)$ into itself and it is a strict contraction, uniformly for x_0 in bounded subsets of X.

Let us now show that $\|(U(x))(t) - T(t)x_0\|_X \leqslant \mu$, for all $t \in (0, \tau_0]$. First note that

$$t^{1-\alpha}\|x(t)\|_X \leqslant t^{1-\alpha}\|x(t) - T(t)x_0\|_X + t^{1-\alpha}\|T(t)x_0\|_X \leqslant t^{1-\alpha}\mu + C\|x_0\|_X.$$

Hence, if B is a bounded subset of X,

$$k = \sup_{x_0\in B} \sup_{x\in K(\tau_0,x_0)} \sup\{\theta^{1-\alpha}\|x(\theta)\|_X : s \in (0, \tau_0]\}$$

$$\begin{aligned}
\|(U(x))(t) - T(t)x_0\|_X &\leqslant C \int_0^t (t-\theta)^{\beta-1} \|f(x(\theta))\|_Y \, d\theta \\
&\leqslant c\,C \int_0^t (t-\theta)^{\beta-1} (1 + \|x(\theta)\|_X^\rho) \, d\theta \\
&\leqslant \frac{c\,C}{\beta}\, t^\beta + c\,C \int_0^t (t-\theta)^{\beta-1} \theta^{-\rho(1-\alpha)} (\theta^{1-\alpha} \|x(\theta)\|_X)^\rho \, d\theta \\
&\leqslant \frac{c\,C}{\beta}\, t^\beta + c\,C \int_0^t (t-\theta)^{\beta-1}\, \theta^{-\rho(1-\alpha)} \, d\theta \; k^\rho \\
&\leqslant \frac{c\,C}{\beta}\, t^\beta + c\,C k^\rho t^{\beta-\rho(1-\alpha)} \int_0^1 (1-\theta)^{\beta-1}\, \theta^{-\rho(1-\alpha)} \, d\theta \\
&\leqslant \frac{c\,C}{\beta}\, t^\beta + c\,C k^\rho t^{\beta-\rho(1-\alpha)} \mathcal{B}(\beta, 1-\rho(1-\alpha)) \\
&\leqslant \mu,
\end{aligned}$$

for suitably small τ_0 and for all $x_0 \in B$, where $\mathcal{B}$ denotes the Beta function; i.e. $\mathcal{B}(a,b) = \int_0^1 r^{a-1}(1-r)^{b-1} dr$ for $a, b > 0$. In particular, for each $x \in K(\tau_0, x_0)$ we have that

$$\|(U(x))(t) - T(t)x_0\|_X \to 0, \quad \text{as } t \to 0. \tag{2.13}$$

Hence, with this choice of τ_0, U takes $K(\tau_0, x_0)$ into itself for any $x_0 \in B$.

Furthermore,

$$\begin{aligned}
\|(U(x))(t) - (U(y))(t)\|_X &\leqslant C \int_0^t (t-\theta)^{\beta-1} \|f(x(\theta)) - f(y(\theta))\|_Y \, d\theta \\
&\leqslant c\,C \int_0^t (t-\theta)^{\beta-1} (1 + \|x(\theta)\|_X^{\rho-1} + \|y(\theta)\|_X^{\rho-1}) \|x(s) - y(s)\|_X \, d\theta \\
&\leqslant \left(\frac{c\,C}{\beta}\, t^\beta + 2c\,C \int_0^t (t-\theta)^{\beta-1}\, \theta^{-(\rho-1)(1-\alpha)} \, d\theta \; k^{\rho-1} \right) \|x-y\|_{K(\tau_0,x_0)} \\
&\leqslant \left(\frac{c\,C}{\beta}\, t^\beta + 2c\,C k^{\rho-1} t^{\beta-(\rho-1)(1-\alpha)} \mathcal{B}(\beta, 1-(\rho-1)(1-\alpha)) \right) \|x-y\|_{K(\tau_0,x_0)} \\
&\leqslant \frac{1}{2} \|x-y\|_{K(\tau_0,x_0)}
\end{aligned} \tag{2.14}$$

for suitably small τ_0 and uniformly with respect to $x_0 \in B$ (here we have used that $\rho < \frac{\beta}{1-\alpha}$). After this, we have that U takes $K(\tau_0, x_0)$ into itself and it is a contraction uniformly with respect to $x_0 \in B$. It follows form the Banach contraction principle that U has a unique fixed point in $K(\tau_0, x_0)$. Hence, the initial value problem (2.9) has a unique solution in the sense of Definition 2.9.

As for the continuity relatively to initial condition, it follows that

$$
\begin{aligned}
&\|x(t,x_0)-y(t,y_0)-T(t)(x_0-y_0)\|_X \leqslant C\int_0^t (t-\theta)^{\beta-1}\|f(x(\theta,x_0))-f(y(\theta,y_0))\|_Y\,d\theta\\
&\leqslant c\,C\int_0^t (t-\theta)^{\beta-1}(1+\|x(\theta,x_0)\|_X^{\rho-1}+\|y(\theta,y_0)\|_X^{\rho-1})\|x(\theta,x_0)-y(\theta,y_0)-T(\theta)(x_0-y_0)\|_X\,d\theta\\
&+c\,C\int_0^t (t-\theta)^{\beta-1}(1+\|x(\theta,x_0)\|_X^{\rho-1}+\|y(\theta,y_0)\|_X^{\rho-1})\|T(\theta)(x_0-y_0)\|_X\,d\theta\\
&\leqslant c\,C\int_0^t (t-\theta)^{\beta-1}\|x(\theta,x_0)-y(\theta,y_0)-T(\theta)(x_0-y_0)\|_X\,d\theta\\
&+c\,C\int_0^t (t-\theta)^{\beta-1}\theta^{(\rho-1)(\alpha-1)}2k^{\rho-1}\|x(\theta,x_0)-y(\theta,y_0)-T(\theta)(x_0-y_0)\|_X\,d\theta\\
&+c\,C\int_0^t (t-\theta)^{\beta-1}\left(\theta^{\alpha-1}+2k^{\rho-1}\theta^{\rho(\alpha-1)}\right)\theta^{1-\alpha}\|T(\theta)(x_0-y_0)\|_X\,d\theta\\
&\leqslant \left(\frac{c\,C}{\beta}\,t^{\beta}+2c\,C\int_0^t (t-\theta)^{\beta-1}\,\theta^{-(\rho-1)(1-\alpha)}\,d\theta\;k^{\rho-1}\right)\zeta(\tau_0)\\
&+c\,C^2\int_0^t (t-\theta)^{\beta-1}\left(\theta^{\alpha-1}+2k^{\rho-1}\theta^{\rho(\alpha-1)}\right)\,d\theta\;\|x_0-y_0\|_X
\end{aligned}
$$

where $\zeta(\tau_0)=\sup_{\theta\in(0,\tau_0]}\|x(\theta,x_0)-y(\theta,y_0)-T(\theta)(x_0-y_0)\|_X$. Hence,

$$
\begin{aligned}
\zeta(\tau_0) \leqslant{}& \left(\frac{c\,C}{\beta}\,\tau_0^{\beta}+c\,C2k^{\rho-1}\tau_0^{\beta-(\rho-1)(1-\alpha)}\mathcal{B}(\beta,1-(\rho-1)(1-\alpha))\right)\zeta(\tau_0)\\
&+c\,C^2\int_0^{\tau_0}(\tau_0-\theta)^{\beta-1}\left(\theta^{\alpha-1}+2k^{\rho-1}\theta^{\rho(\alpha-1)}\right)\,d\theta\;\|x_0-y_0\|_X
\end{aligned}
$$

and as a consequence of that, if τ_0 is suitably small,

$$\sup_{t\in(0,\tau_0]}\|x(t)-y(t)-T(t)(x_0-y_0)\|_X \leqslant C\|x_0-y_0\|_X.$$

This is saying that the solutions of the semilinear problem (2.9) behave exactly as the solutions of the corresponding linear problem, also with respect to initial conditions, that is

$$\|x(t)-y(t)\|_X \leqslant Ct^{\alpha-1}\|x_0-y_0\|_X,\ t\in(0,\tau_0].$$

We note that the above continuity with respect to initial conditions is uniform in bounded subsets B of X; that is,

$$\sup_{x_0\in B}\sup_{t\in(0,\tau_0]}\|x(t,x_0+h_0)-x(t,x_0)-T(t)h_0\|_X \leqslant C_B\|h_0\|_X.$$

Next we observe that the continuation of solutions holds in the following sense, if a solution defined on its maximal interval of existence $x(\cdot,x_0):(0,\tau_{\max})$, then either $\tau_{\max}=+\infty$ or $\limsup_{t\to\tau_{\max}}\|x(t,x_0)\|_X=+\infty$. This is accomplished simply noting that the choice of τ_0 in the proof of existence can be made uniform in bounded subsets of X. ∎

3. The Linear Operator Associated to (1.3)

In this section we consider the evolution problem (1.3) and analyze the structure of the linear elliptic and parabolic problem. We will see that the linear operator associated to (1.3) presents the deficiency in the resolvent estimate as explained in the previous section, see 2.3. Therefore, we will be able to apply the results on generation of semigroups and existence and uniqueness results for the semilinear parabolic problem, Proposition 2.11.

Consider the Banach space U_0^p defined in the introduction, and let $\Lambda_0 : D(\Lambda_0) \subset U_0^p \to U_0^p$ be the operator defined by

$$D(\Lambda_0) = \big\{(w,v) \in U_0^p : w \in D(\Delta_N^\Omega),\ (gv')' \in L^p(0,1),\ v(0) = w(P_0),\ v(1) = w(P_1)\big\}$$
$$\Lambda_0(w,v) = \Big(-\Delta w + (\mu + W(x))w, -\frac{1}{g}(gv')' + (\mu + V(s))v\Big), \quad (w,v) \in D(\Lambda_0), \tag{3.1}$$

where Δ_N^Ω is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$ with $D(\Delta_N^\Omega) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\}$, $(W,V) \in L^\infty(\Omega) \oplus L^\infty(0,1)$ and $\mu + W(x) \geqslant 1$ for all $x \in \Omega$, $\mu + V(s) \geqslant 1$ for all $s \in [0,1]$. Moreover, since we are assuming that g is a Lipschitz function, we have that $D(\Lambda_0) \subset W^{2,p}(\Omega) \times W^{2,p}(0,1)$. Hence, for $p > N/2$ we have that $D(\Delta_N^\Omega)$ is continuously embedded in $C(\bar{\Omega})$. This tells us that the functions in $D(\Delta_N^\Omega)$ have trace at P_0 and P_1.

Note that, if $\mu + W(x) \equiv 1$, $\mu + V(s) \equiv 1$ and A_0 is the operator defined in the introduction, we have that $\Lambda_0 = A_0$.

Proposition 3.1. *The operator Λ_0 defined by (3.1) has the following properties*
(i) $D(\Lambda_0)$ is dense in U_0^p,
(ii) Λ_0 is a closed operator,
(iii) Λ_0 has compact resolvent and
(iv) $\rho(\Lambda_0) \supset \Sigma_\theta$, where Σ_θ is given by (2.2) and, for $\frac{N}{2} < q \leqslant p$, we have the following estimates

$$\|(\Lambda_0 + \lambda)^{-1}\|_{\mathcal{L}(U_0^q, U_0^p)} \leqslant \frac{C}{|\lambda|^\alpha + 1} \tag{3.2}$$

and

$$\|\Lambda_0(\Lambda_0 + \lambda)^{-1}\|_{\mathcal{L}(U_0^p)} \leqslant C(1 + |\lambda|^{1-\tilde{\alpha}}). \tag{3.3}$$

for each $0 < \alpha < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) < 1$, $0 < \tilde{\alpha} < 1 - \frac{N}{2p} < 1$ and $\lambda \in \Sigma_\theta$.
(v) If B_0 is the realization of Λ_0 in $C(\bar{\Omega}) \oplus L^p(0,1)$ we have that B_0 is a sectorial operator in $C(\bar{\Omega}) \oplus L_g^p(0,1)$ with compact resolvent. Therefore $-B_0$ generates an strongly continuous, analytic semigroup $e^{-B_0 t}$ in $C(\bar{\Omega}) \oplus L_g^p(0,1)$.

Proof: *(i)* Let $(w,v) \in L^p(\Omega) \oplus L_g^p(0,1)$. Let $(w_n, v_n) \in C_0^\infty(\Omega) \oplus C_0^\infty(0,1)$ with $(w_n, v_n) \to (w,v)$ in $L^p(\Omega) \oplus L_g^p(0,1)$, then $(w_n, v_n) \in D(\Lambda_0)$ and the result is proved.
(ii) Let $(w_n, v_n) \in D(\Lambda_0)$ be such that $(w_n, v_n) \to (w,v)$ and $\Lambda_0(w_n, v_n) \to (\phi, \psi)$ in $L^p(\Omega) \oplus L_g^p(0,1)$. Since $w_n \in D(\Delta_N^\Omega)$ and Δ_N^Ω is a closed operator in $L^p(\Omega)$, see [11], we have that $w \in D(\Delta_N^\Omega)$ and $w_n \to w$ in $W^{2,p}(\Omega)$. In particular, $-\Delta w_n \to -\Delta w$ and since $p > N/2$ we have $W^{2,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, which implies that $w_n(P_0) \to w(P_0)$ and $w_n(P_1) \to w(P_1)$. On the

other hand $v_n \to v$ and $\psi_n = -\dfrac{1}{g}(g\,v_n')' + (\mu + V)v_n \to \psi$ in $L^p_g(0,1)$. Now

$$\begin{cases} -\dfrac{1}{g}(g\,v_n')' + (\mu + V(s))v_n = \psi_n, & s \in (0,1) \\ v_n(0) = w_n(P_0), \quad v_n(1) = w_n(P_1). \end{cases}$$

Making the change of variables $z_n = v_n - \xi_n$, where ξ_n is the solution of the following problem

$$\begin{cases} -\dfrac{1}{g}(g\,\xi_n')' = 0, & s \in (0,1) \\ \xi_n(0) = w_n(P_0), \quad \xi_n(1) = w_n(P_1), \end{cases} \tag{3.4}$$

we have

$$\begin{cases} -\dfrac{1}{g}(g z_n')' + (\mu + V(s))z_n = \psi_n - (\mu + V(s))\xi_n, & s \in (0,1) \\ z_n(0) = z_n(1) = 0. \end{cases}$$

Using the linearity of problem (3.4), it is easy to see that $\xi_n(s) = w_n(P_0)\chi^{(1,0)}(s) + w_n(P_1)\chi^{(0,1)}(s)$, where $\chi^{(a,b)}(s)$ is the unique solution of

$$\begin{cases} -\dfrac{1}{g}(g\chi')' = 0, & s \in (0,1) \\ \chi(0) = a, \quad \chi(1) = b. \end{cases} \tag{3.5}$$

Moreover, direct integration, shows that

$$\chi^{(1,0)}(s) = \frac{\displaystyle\int_s^1 \frac{1}{g(\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{g(\theta)}d\theta} \qquad \chi^{(0,1)}(s) = \frac{\displaystyle\int_0^s \frac{1}{g(\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{g(\theta)}d\theta} \tag{3.6}$$

Hence,

$$\xi_n(s) = w_n(P_0)\frac{\displaystyle\int_s^1 \frac{1}{g(\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{g(\theta)}d\theta} + w_n(P_1)\frac{\displaystyle\int_0^s \frac{1}{g(\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{g(\theta)}d\theta} \tag{3.7}$$

and, since $w_n(P_0) \to w(P_0)$, $w_n(P_1) \to w(P_1)$, it follows that $\xi_n \to \xi$, where ξ is the solution of the following problem

$$\begin{cases} -\dfrac{1}{g}(g\,\xi')' = 0, & s \in (0,1) \\ \xi(0) = w(P_0), \quad \xi(1) = w(P_1). \end{cases} \tag{3.8}$$

Moreover, since the operator $\mathcal{L}(v) = -\frac{1}{g}(gv')'$ with homogeneous Dirichlet boundary conditions at $s = 0$ and $s = 1$ is closed in $L^p_g(0,1)$, we have that $z_n \to z$ in $L^p_g(0,1)$ where z

satisfies

$$\begin{cases} -\dfrac{1}{g}\,(g\,z')' + (\mu + V(s))z = \psi - (\mu + V(s))\xi, \quad s \in (0,1) \\ z(0) = z(1) = 0. \end{cases}$$

From which it follows that $v_n = z_n + \xi_n \to z + \xi = v$, and v satisfies

$$\begin{cases} -\dfrac{1}{g}\,(g\,v')' + (\mu + V(s))v = \psi, \quad s \in (0,1) \\ v(0) = w(P_0),\ v(1) = w(P_1). \end{cases} \tag{3.9}$$

which shows that Λ_0 is closed
(iii) Since $D(\Lambda_0) \subset W^{2,p}(\Omega) \oplus W^{2,p}(0,1) \hookrightarrow L^p(\Omega) \oplus L^p_g(0,1)$ and since the embedding $W^{2,p}(\Omega) \oplus W^{2,p}(0,1) \hookrightarrow L^p(\Omega) \oplus L^p_g(0,1)$ is compact, it follows that Λ_0 has compact resolvent.
(iv) Let $(f,h) \in U^p_0$. Solving the equation $(w,v) = (\Lambda_0 + \lambda)^{-1}(f,h)$ is equivalent to solve $(\Lambda_0 + \lambda)(w,v) = (f,h)$, which is equivalent to find the functions (w,v) verifying,

$$\begin{cases} -\Delta w + (\mu + W(x))w + \lambda w = f, & x \in \Omega \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ & \\ -\dfrac{1}{g}\,(g\,v')' + (\mu + V(s))v + \lambda v = h, & s \in (0,1), \\ v(0) = w(P_0), \quad v(1) = w(P_1). \end{cases} \tag{3.10}$$

Using the resolvent estimates for the Laplace operator with homogeneous Neumann boundary conditions we obtain

$$\left.\begin{aligned} \|w\|_{L^p(\Omega)} &\leqslant \frac{C}{|\lambda|^{1-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} + 1}\|f\|_{L^q(\Omega)}, \\ \|w\|_{H^{r,q}(\Omega)} &\leqslant \frac{C}{|\lambda|^{1-r/2} + 1}\|f\|_{L^q(\Omega)} \end{aligned}\right\} \lambda \in \Sigma_\theta \tag{3.11}$$

where we recall that $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leqslant \pi - \theta\}$.

We consider now the change of variables $z = v - \xi$, where ξ is the solution of (3.8), and we apply it to the last two equations of (3.10), we have

$$\begin{cases} -\dfrac{1}{g}\,(g\,z')' + (\mu + V(x))z + \lambda z = h - \xi - \lambda\xi, \quad s \in (0,1) \\ z(0) = z(1) = 0. \end{cases}$$

Note that, if $A_g : D(A_g) \subset L^p_g(0,1) \to L^p_g(0,1)$ is the operator given by

$$\begin{aligned} D(A_g) &= \{z \in L^p_g(0,1) : (gz')' \in L^p_g(0,1) : z(0) = z(1) = 0\} \\ A_g z &= -\frac{1}{g}\,(g\,z')' + (\mu + V(x))z,\ \forall z \in D(A_g), \end{aligned}$$

we have the following resolvent estimates

$$\left.\|(A_g+\lambda)^{-1}y\|_{L^p_g(0,1)} \leqslant \frac{C}{|\lambda|^{1-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}+1}\|y\|_{L^q_g(0,1)}\right\},\ \forall \lambda \in \Sigma_\theta \tag{3.12}$$

Applying (3.12) to $y = h-(\lambda+1)\xi$ we get

$$\begin{aligned}\|z\|_{L^p_g(0,1)} &\leqslant \frac{C}{|\lambda|^{1-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}+1}\|h-(\lambda+1)\xi\|_{L^q_g(0,1)} \\ &\leqslant \frac{C}{|\lambda|^{1-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}+1}\|h\|_{L^q_g(0,1)} + \tilde{C}|\lambda|^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\xi\|_{L^q_g(0,1)}\end{aligned} \tag{3.13}$$

Hence, for $v = z+\xi$ we have

$$\|v\|_{L^p_g(0,1)} \leqslant \frac{C}{|\lambda|^{1-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}+1}\|h\|_{L^q_g(0,1)} + \tilde{C}|\lambda|^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\xi\|_{L^q_g(0,1)} + \|\xi\|_{L^p_g(0,1)}.$$

But notice that $\|\xi\|_{L^q_g(0,1)} \leqslant C\|\xi\|_{L^p_g(0,1)} \leqslant C(|w(P_0)|+|w(P_1)|) \leqslant C\|w\|_{C^0(\bar{\Omega})}$, and using the embedding $W^{r,q}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ for any $r > N/q$, we have that

$$\|\xi\|_{L^q_g(0,1)} \leqslant C\|\xi\|_{L^p_g(0,1)} \leqslant C\|w\|_{W^{r,q}(\Omega)} \tag{3.14}$$

Hence, from (3.11),

$$\begin{aligned}\|v\|_{L^p_g(0,1)} &\leqslant \frac{C}{|\lambda|^{1-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}+1}\|h\|_{L^q_g(0,1)} + (\tilde{C}+1)|\lambda|^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|w\|_{W^{r,q}(\Omega)} \\ &\leqslant \frac{C}{|\lambda|^{1-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}+1}\|h\|_{L^q_g(0,1)} + \frac{C}{|\lambda|^{1-\frac{1}{2}(\frac{1}{p}-\frac{1}{q}-r)}+1}\|f\|_{L^q(\Omega)}.\end{aligned}$$

This concludes the proof of (3.2).
(v) To prove that B_0 is sectorial we proceed exactly as in (3.11) changing $L^p(\Omega)$ by $C(\bar{\Omega})$, noting that $\|\xi\|_{L^p(0,1)} \leqslant C\|w\|_{C(\bar{\Omega})}$ and $|\lambda|\|w\|_{C(\bar{\Omega})} \leqslant C\|f\|_{C(\bar{\Omega})}$. ■

Remark 3.2. *From estimates (3.2), (3.3) and realizing that* $D(A_0) \hookrightarrow W^{2,p}(\Omega) \oplus W^{2,p}(0,1)$, *we get that*

$$\|(A_0+\lambda)^{-1}\|_{\mathcal{L}(U^p_0,U^p_0)} \le \frac{C}{|\lambda|^{\tilde{\alpha}}+1}$$

$$\|(A_0+\lambda)^{-1}\|_{\mathcal{L}(U^p_0,W^{2,p}(\Omega)\oplus W^{2,p}(0,1))} \le C(|\lambda|^{1-\tilde{\alpha}}+1)$$

Interpolating both inequalities, we get

$$\|(A_0+\lambda)^{-1}\|_{\mathcal{L}(U^p_0,W^{1,p}(\Omega)\oplus W^{1,p}(0,1))} \le \frac{C}{|\lambda|^{\tilde{\alpha}-\frac{1}{2}}+1}$$

where $0 < \tilde{\alpha} < 1-\frac{N}{2p} < 1$. *Noticing that* $\tilde{\alpha} \to 1$ *as* $p \to +\infty$, *we may choose* p *large enough so that* $\tilde{\alpha}-\frac{1}{2} > 0$.

Moreover, with the expression of the linear semigroup in terms of the integral of the resolvent operator,

$$e^{A_0 t} = \frac{1}{2\pi i}\int_\Gamma (A_0+\lambda)^{-1}e^{\lambda t}d\lambda$$

we easily get

$$\|e^{A_0 t}\|_{\mathcal{L}(U_0^p, W^{1,p}(\Omega)\oplus W^{1,p}(0,1))} \leq Ct^{-\beta}$$

with $\beta = \frac{3}{2} - \tilde{\alpha} < 1$.

3.1. **Singularity of the semigroup at $t = 0$.** Notice that Proposition 3.1 ensures that the resolvent estimate (2.3) holds for the operator A_0 and, from Proposition 2.1, that the semigroup $T(t)$ associated to it satisfies (2.5), that is $\|T(t)\|_{\mathcal{L}(U_0^p)} \leqslant Ct^{-1+\alpha}$, with $0 < \alpha < 1 - \frac{N}{2p}$, and therefore we are not able to show that the semigroup $T(t)$ is continuous, nor even bounded, as $t \to 0^+$. We show now that, actually, the semigroup is not continuous at $t = 0$. We will prove that this singularity property at $t = 0$ is unremovable in the case $p = 2$. For this, we will choose an initial condition u_0 which lies in U_0^2 and show that for this initial condition $\|T(t)u_0\|_{U_0^2} \geqslant ct^{-\delta}$ for some positive constants c and δ.

As a matter of fact we will choose $0 < \alpha < N/2$ and consider the radially symmetric function

$$w_0(x) = \begin{cases} |x|^{-\alpha}, & x \in B(P_0, \rho/2) \\ 0, & x \in \mathbb{R}^N \setminus B(P_0, \rho/2), \end{cases} \tag{3.15}$$

with $\rho > 0$ small enough with the property that $B(P_0, \rho) \cap \Omega = \{x \in B(P_0, \rho) : x_1 < 0\}$. Recall that $P_0 = (0, 0, \ldots, 0)$ and $P_1 = (1, 0, \ldots, 0)$ are the points of junction of Ω with the line segment R_0. Moreover, we will assume that the set Ω is given by the union of two disconnected domains, one at the left, Ω_0^L and the other at the right, Ω_0^R, just as it has been depicted in Figure 2. Moreover, to simplify the analysis we will assume that the function g appearing in the differential equation in the line segment, satisfies $g \equiv 1$.

Since $0 < \alpha < N/2$, we have that $w_0 \in L^2(\mathbb{R}^N)$.

Lemma 3.3. *For the initial condition w_0 above, the solution $\tilde{w}(t,x)$ of the problem*

$$\begin{cases} \tilde{w}_t = \Delta \tilde{w}, \quad \Omega, \ t > 0 \\ \dfrac{\partial \tilde{w}}{\partial n} = 0, \quad \partial\Omega \\ \tilde{w}(0,x) = w_0(x), \quad \partial\Omega \end{cases} \tag{3.16}$$

satisfies

$$0 < c_1 t^{-\alpha/2} \leqslant \tilde{w}(t, P_0) \leqslant c_2 t^{-\alpha/2}, \quad 0 < t < t_0$$

for some constants $c_1, c_2 > 0$ and for some $t_0 > 0$ small.

Proof: If we consider the solution of $U_t = \Delta U$ in $\mathbb{R}^N$ with initial condition $U(0,x) = w_0(x)$, we know that this solution is given by the convolution with the heat kernel, that is,

$$U(t,x) = \int_{\mathbb{R}^N} K(t, x-y) w_0(y) dy$$

where $K(t,x) = (4\pi t)^{-N/2} \exp(-|x|^2/4t)$.

Notice also that by symmetry, this function U is also the solution of $u_t = \Delta u$ in $\mathbb{R}^N_- = \{x \in \mathbb{R}^N, x_1 < 0\}$, with Neumann boundary conditions in $\{x_1 = 0\}$, that is $\frac{\partial u}{\partial x_1} = 0$ in $\{x_1 = 0\}$.

But if we evaluate the behavior of $U(t,x)$ at $x=0$, we get

$$U(t,0)=\int_{\mathbb{R}^N}(4\pi t)^{-N/2}\exp(-|y|^2/4t)w_0(y)dy=(4\pi t)^{-N/2}\int_{|y|<\rho/2}\exp(-|y|^2/4t)|y|^{-\alpha}dy.$$

Changing to polar coordinates and with the appropriate changes of variables, we get

$$U(t,0)=C(N)t^{-N/2}\int_0^{\rho/2}\exp(-r^2/4t)r^{-\alpha}r^{N-1}dr=C(N)t^{-\alpha/2}\int_0^{\frac{\rho}{2\sqrt{t}}}\exp(-s^2/4)s^{N-\alpha-1}ds$$

This last statement implies that we can choose two constants $\tilde{c}_1,\tilde{c}_2>0$, such that

$$\tilde{c}_1 t^{-\alpha/2}\leqslant U(t,0)\leqslant \tilde{c}_2 t^{-\alpha/2},\quad \forall t\in(0,1)$$

By elliptic and parabolic regularity results we easily get that there exists a constant m such that $|\tilde{w}(t,x)|\leqslant m$ for all $x\in\Omega\setminus B(P_0,\rho/2)$ and for all $t\in(0,1)$. Hence, it is not difficult to see from comparison arguments that $\tilde{w}(t,x)\leqslant U(t,x)+m$. Similarly if $|U(t,x)|\leqslant M$ for all $x\in\Omega\setminus B(P_0,\rho/2)$, then $U(t,x)\leqslant\tilde{w}(t,x)+M$. This implies that there exists two constants $M,m>0$ such that

$$\tilde{c}_1 t^{-\alpha/2}-M\leqslant\tilde{w}(t,P_0)\leqslant\tilde{c}_2 t^{-\alpha/2}+m,\quad\forall t\in(0,1)$$

which implies that there exists $t_0>0$ and constants c_1,c_2 such that

$$c_1 t^{-\alpha/2}\leqslant\tilde{w}(t,P_0)\leqslant c_2 t^{-\alpha/2},\quad\forall t\in(0,t_0).$$

which shows the lemma. ■

Remark 3.4. *Observe that since $\Omega=\Omega_0^L\cup\Omega_0^R$ and we are assuming that Ω_0^L, Ω_0^R are disjoint, then $\tilde{w}(t,P_1)\equiv 0$.*

We consider now the solution of the following problem in the line segment $R_0\equiv(0,1)$,

$$\begin{cases}\tilde{v}_t-\tilde{v}_{xx}=0, & x\in(0,1)\\ \tilde{v}(t,0)=\tilde{w}(t,P_0),\ \tilde{v}(1)=\tilde{w}(t,P_1)\equiv 0 & \\ \tilde{v}(0,x)=0, & x\in(0,1).\end{cases}\tag{3.17}$$

We can prove

Lemma 3.5. *We have that $\|\tilde{v}(t,\cdot)\|_{L^2(0,1)}\geqslant Ct^{\frac{1}{2}-\alpha}$ for $0<t<t_0$ for some small $t_0>0$.*

Proof: The solution of (3.17) is given by

$$\tilde{v}(t,\cdot)=\int_0^t\tilde{w}(s,P_0)Le^{L(t-s)}\chi ds\tag{3.18}$$

where the function $\chi(x)=1-x$ and the operator L is the unbounded operator in $L^2(0,1)$ defined by $Lu=u_{xx}$ with homogeneous Dirichlet boundary conditions. Expression (3.18) is obtained by the change of variables in (3.17) given by $z(t,x)=\tilde{v}(t,x)-\tilde{w}(t,P_0)\chi(x)$, applying the variation of constanst formula to the equation satisfied by z and undoing the change of variables in the variation of constants formula.

We analyze now (3.18) using the spectral decomposition of the opeator L. If we denote by λ_k and φ_k the eigenvalues and eigenfunctions of $-L$, that is $\lambda_k = \pi^2 k^2$ and $\varphi_k(x) = \sin(\pi k x)/\sqrt{2}$, we have

$$\|\tilde{v}(t,\cdot)\|^2_{L^2(0,1)} = \sum_{k=1}^{\infty} (\chi, \varphi_k)^2 \lambda_k^2 (\int_0^t \tilde{w}(s,P_0) e^{-\lambda_k(t-s)} ds)^2.$$

But,

$$\int_0^t \tilde{w}(s,P_0) e^{-\lambda_k(t-s)} ds \geqslant c_1 \int_0^t s^{-\alpha/2} e^{-\lambda_k(t-s)} ds = c_1 t^{1-\alpha/2} \int_0^1 z^{-\alpha/2} e^{-\lambda_k(1-z)t} dz$$

Separating the last integral in two parts, one from 0 to 1/2 and the other from 1/2 to 1, so that we isolate each singularity of the integrands and performing elementary integrations, we obtain that

$$\int_0^1 z^{-\alpha/2} e^{-\lambda_k(1-z)t} dz \geqslant \frac{C}{\lambda_k t}(1 - e^{-\lambda_k t/2})$$

for some constant C independent of k and $t > 0$. Therefore

$$\|\tilde{v}(t,\cdot)\|^2_{L^2(0,1)} \geqslant C t^{-\alpha} \sum_{k=1}^{\infty} (\chi, \varphi_k)^2 (1 - e^{-\lambda_k t/2})^2$$

But, $(\chi, \varphi_k)^2 \geqslant C k^{-2}$ and $(1 - e^{-\lambda_k t/2})^2 = (1 - e^{-t\pi^2 k^2/2})^2$

Hence, we need to estimate the behavior as $t \to 0$ of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}(1 - e^{-k^2 t})^2$, which is the same as the behavior of the improper integral, $\int_1^{\infty} \frac{1}{x^2}(1 - e^{-x^2 t})^2 dx$. Changing variables, $z = x\sqrt{t}$, we get,

$$\int_1^{\infty} \frac{1}{x^2}(1 - e^{-x^2 t})^2 dx = t^{1/2} \int_{\sqrt{t}}^{\infty} \frac{1}{z^2}(1 - e^{-z^2})^2 dz \geqslant C t^{1/2}$$

where we use that

$$0 < \int_0^{\infty} \frac{1}{z^2}(1 - e^{-z^2})^2 dz < +\infty.$$

Putting all the information together we get,

$$\|\tilde{v}(t,\cdot)\|^2_{L^2(0,1)} \geqslant C t^{\frac{1}{2}-\alpha}, \text{ for } 0 < t < t_0$$

with $t_0 > 0$ small enough, which shows the lemma. ■

With these two lemmas, we can show now that the semigroup generated by A_0 is not continuous at $t = 0$ in L^2. We have the following

Proposition 3.6. *If we consider the initial condition $(w_0, 0) \in L^2(\Omega) \times L^2(0,1)$ where w_0 is the function from Lemma 3.3 then, if $T(t)$ is the semigroup generated by the operator A_0,* *that is* $(w(t,\cdot), v(t,\cdot)) = T(t)(w_0, 0)$ *is the solution of*

$$\begin{cases} w_t - \Delta w + w = 0, \quad x \in \Omega,\ t > 0 \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \\ w(0,x) = w_0(x) \\ v_t - v_{xx} + v = 0, \quad x \in (0,1) \\ v(0) = w(P_0),\ v(1) = w(P_1) \\ v(0,x) = 0 \end{cases} \tag{3.19}$$

then

$$\|T(t)(w_0,0)\|_{U^2} \geqslant Ct^{\frac{1}{2}-\alpha} \to +\infty \text{ as } t \to 0$$

Proof: The proof is simple now. We just need to realize that with the appropriate and standard change of variables, we have that $(w(t,x), v(t,x)) = e^{-t}(\tilde{w}(t,x), \tilde{v}(t,x))$, where $\tilde{w}$ and $\tilde{v}$ are the solutions of (3.16) and (3.17), respectively. Applying Lemma 3.5 we obtain the result. ■

3.2. **The Eigenvalue Problem.** Next we analyze in detail the spectrum of the linear operator A_0, see (3.1) and to simplify and since we want to perform explicit computations, we will consider that the potentials W, V are identically zero. That is, $A_0(w,v) = (-\Delta w + w, -\frac{1}{g}(gv')' + v)$. Observe that, from Proposition 3.1, the operator A_0 has compact resolvent. In particular, its spectrum consists only of eigenvalues. We will see that all the eigenvalues of A_0 are positive real numbers but nevertheless we will show that there are eigenvalues for which the geometric and algebraic multiplicity do not coincide. This is another special feature of this operator and indicates that A_0 does not have a *self adjoint structure* in U_0^p.

We wish to find λ and $(w,v) \neq (0,0)$ such that

$$A_0(w,v) - \lambda(w,v) = 0. \tag{3.20}$$

The equation (3.20) is equivalent to the system

$$\begin{cases} -\Delta w + w = \lambda\, w, \quad x \in \Omega \\ \quad \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \end{cases} \tag{3.21}$$

$$\begin{cases} -\dfrac{1}{g}\,(gv_x)_x + v = \lambda\, v, \quad s \in (0,1), \\ \quad v(0) = w(P_0), \quad v(1) = w(P_1). \end{cases} \tag{3.22}$$

We denote by $\{\mu_i\}_{i=1}^{\infty}$ the eigenvalues of (3.21), ordered and counting multiplicity and by $\{\phi_i\}_{i=1}^{\infty}$ a corresponding set of orthonormal eigenfunctions (orthonormal in the sense of $L^2(\Omega)$). We also consider the following eigenvalue problem

$$\begin{cases} -\dfrac{1}{g}\,(gv_x)_x + v = \tau\, v, \quad s \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{3.23}$$

and we denote by $\{\tau_i\}_{i=1}^{\infty}$, its eigenvalues, ordered and counting multiplicity and by $\{\gamma_i\}_{i=1}^{\infty}$ a corresponding set of orthonormal eigenfunctions. We have the following,

Proposition 3.7. $\sigma(A_0) = \{\mu_i\}_{i=1}^{\infty} \cup \{\tau_i\}_{i=1}^{\infty}$.

Proof: First we show the inclusion $\sigma(A_0) \subset \{\mu_i\}_{i=1}^{\infty} \cup \{\tau_i\}_{i=1}^{\infty}$. In fact, if $\lambda \in \sigma(A_0)$, then there exists $(w, v) \neq 0$ such that (3.21), (3.22) hold, and therefore we have the following cases:

• If $w = 0$, then $v \neq 0$ and the boundary conditions in (3.22) are $v(0) = v(1) = 0$. Therefore $\lambda \in \{\tau_i\}_{i=1}^{\infty}$.

• If $w \neq 0$, then necessarilly $\lambda \in \{\mu_i\}_{i=1}^{\infty}$.

Hence, $\lambda \in \{\mu_i\}_{i=1}^{\infty} \cup \{\tau_i\}_{i=1}^{\infty}$.

For the inclusion $\{\mu_i\}_{i=1}^{\infty} \cup \{\tau_i\}_{i=1}^{\infty} \subset \sigma(A_0)$ we analyze the following cases:

If $\lambda = \tau_i$ then $(w, v) = (0, \gamma_i) \neq (0, 0)$ is the solution to (3.20). Then $\lambda \in \sigma(A_0)$.

If $\lambda \in \{\mu_i\}_{i=1}^{\infty} \setminus \{\tau_i\}_{i=1}^{\infty}$ then $(w, v) = (\phi_i, \chi) \neq (0, 0)$ is solution to (3.20), where χ is the solution of the following problem

$$\begin{cases} -\dfrac{1}{g}\,(g\chi_x)_x + \chi = \lambda\, \chi, \quad x \in (0,1), \\ \chi(0) = \phi(P_0), \quad \chi(1) = \phi(P_1), \end{cases} \tag{3.24}$$

which it is not difficult to see, by the Fredholm alternative, that it will always exist since $\lambda \notin \{\tau_i\}_{i=1}^{\infty}$. Thus, $\lambda \in \sigma(A_0)$. ∎

We want to analyze the multiplicity of the eigenvalues of A_0. Recall that if $\lambda \in \sigma(A_0)$, then the geometric multiplicity of λ is given by $m_g(\lambda) = \dim(N(A_0 - \lambda I))$ and any nonzero function $\varphi \in N(A_0 - \lambda I)$ is an eigenfunction associated to λ. Moreover, it is known that for this λ, there will exists an integer $m \geqslant 1$ such that $\dim(N(A_0 - \lambda I)) < \dim(N(A_0 - \lambda I)^2) < \cdots < \dim(N(A_0 - \lambda I)^m) = \dim(N(A_0 - \lambda I)^{m+1}) = \dim(N(A_0 - \lambda I)^{m+2}) = \ldots$ and the algebraic multiplicity of λ is given by $m_a(\lambda) = \dim(N(A_0 - \lambda I)^m)$.

If an operator is selfadjoint then $m = 1$ and for each eigenvalue the algebraic and geometric multiplicity always coincide.

We will see that for the operator A_0, we may have eigenvalues with $m > 1$ and in particular, $m_g(\lambda) < m_a(\lambda)$.

Proposition 3.8. *Assume that $\lambda \in \sigma(A_0) = \{\mu_i\}_{i=1}^{\infty} \cup \{\tau_i\}_{i=1}^{\infty}$. Then*
i) If $\lambda \in \{\tau_j\} \setminus \{\mu_j\}$, that is $\lambda = \tau_j$ for some j and $\tau_j \notin \{\mu_i\}$, then $m_a(\lambda) = m_g(\lambda) = 1$.
ii) If $\lambda \in \{\mu_j\} \setminus \{\tau_j\}$ and μ_j is an eigenvalue of multiplicity k, then $m_a(\lambda) = m_g(\lambda) = k$.
iii) If $\lambda \in \{\mu_j\} \cap \{\tau_j\}$, that is, $\lambda = \tau_j = \mu_i = \mu_{i+1} = \cdots = \mu_{i+k-1}$, then, $k \leqslant m_a(\lambda) \leqslant m_g(\lambda) = k + 1$. Moreover, there exist numbers α_h, $h = 0, 1 \ldots, k-1$, depending on γ_j, $\phi_i, \ldots, \phi_{i+k-1}$ such that if at least one of them is not 0, then $k = m_a(\lambda) < m_g(\lambda) = k + 1$.

Proof: The proof of i) and ii) is very simple. In i) the unique eigenfunction associated to λ is given by $w = 0$ in Ω nad $v = \gamma_j$ in the segment R. In ii) for each of the functions ϕ_j we find the solution of $-\frac{1}{g}(gv_x)_x + v = \mu_j v$ in R with boundary conditions $v(P_0) = \phi_j(P_0)$, $v(P_1) = \phi_j(P_1)$, which exists and it is unique since $\mu_j \notin \{\tau_i\}$ and it is not difficult to see that these are the unique eigenfunctions.

The proof of iii) is a little more involved. Let us start showing the following:
1) If $(w, v) \in N(A_0 - \lambda I)^m$ with $m \geqslant 1$ then $w \in [\phi_i, \phi_{i+1}, \dots \phi_{i+k-1}]$.
2) If $(0, v) \in N(A_0 - \lambda I)^m$ with $m \geqslant 1$ then $v = c\gamma_j$ for some constant c.
3) For $m \geqslant 2$, we have $N(A_0 - \lambda I)^m = \{(w, v) : (A_0 - \lambda I)(w, v) = (0, c\gamma_j) \text{ for some } c\}$.

To show 1) we use that the first coordinate of A_0 is given by the operator $B_0 = -\Delta + I$ with Neumann boundary conditions which is selfadjoint and it is decoupled from the second coordinate. Hence, if $(w, v) \in N(A_0 - \lambda I)^m$ then $w \in N(B_0 - \lambda I)^m = N(B_0 - \lambda I) = [\phi_i, \phi_{i+1}, \dots \phi_{i+k-1}]$.

To show 2) observe that for $m = 1$, we have that $(A_0 - \lambda I)(0, v) = (0, 0)$ if v is solution of (3.22) with $w(P_0) = 0 = w(P_1)$, which implies that necessarily $v = c\gamma_j$. Moreover, if $m = 2$ and $(A_0 - \lambda I)^2(0, v) = (0, 0)$ and we denote by $(A_0 - \lambda I)(0, v) = (0, v_1)$ then $(0, v_1) \in N(A_0 - \lambda I)$ which implies that $v_1 = c\gamma_j$. Hence, v is a solution of

$$\begin{cases} -\dfrac{1}{g}\,(gv_x)_x + v - \lambda\, v = c\gamma_j, \quad s \in (0,1), \\ v(0) = 0, \quad v(1) = 0. \end{cases} \tag{3.25}$$

But by the Fredholm alternative, this equation has a solution if and onlyif $c = 0$ and the solution is given by $v = \tilde{c}\gamma_j$ for some constan $\tilde{c}$. We easily prove the general result by induction.

To show 3), we realize first that in a trivial way we have that for $m \geqslant 2$, $N(A_0 - \lambda I)^m \supset \{(w, v) : (A_0 - \lambda I)(w, v) = (0, c\gamma_j) \text{ for some } c\}$. Moreover, if $(w, v) \in N(A_0 - \lambda I)^m$, then from 1) we have that $w \in [\phi_i, \dots, \phi_{i+k-1}]$. Therefore, $(A_0 - \lambda I)(w, v) = (0, f) \in N(A_0 - \lambda I)^{m-1}$. But, by 2) $(0, f) = (0, c\gamma_j)$ for some constant c. This proves 3).

Hence, we have obtained that

$$N(A_0 - \lambda I) \subset N(A_0 - \lambda I)^2 = \ldots = N(A_0 - \lambda I)^m = \ldots$$

It is not difficult to see now that $\dim N(A_0 - \lambda I)^2 = k + 1$. For this, note that for any given $w \in [\phi_i, \dots, \phi_{i+k-1}]$, using the Fredholm alternative, there exists a unique constant $c = c(w)$ for which the following problem has solutions

$$\begin{cases} -\dfrac{1}{g}\,(gv_x)_x + v - \lambda\, v = c\gamma_j, \quad s \in (0,1), \\ v(0) = w(P_0), \quad v(1) = w(P_1). \end{cases} \tag{3.26}$$

Moreover, if we denote by V one of this solutions, the set of solutions is given by $V + d\gamma_j$ for any constant d. Hence for fixed w the set of solutions is one dimensional, which implies that $\dim(N(A_0 - \lambda I)^2) = k + 1$.

Let us see now that $\dim(N(A_0 - \lambda I)) \geqslant k$ and in many cases it is k.

If $\lambda = \tau_j = \mu_i = \mu_{i+1} = \cdots = \mu_{i+k-1}$ we look first for $N(A_0 - \lambda)$, that is, we look for all the pairs of functions (w, v) solutions of the problem

$$\begin{cases} -\Delta w + w = \lambda\, w, \quad x \in \Omega \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \end{cases} \tag{3.27}$$

$$\begin{cases} -\dfrac{1}{g}\,(g v_x)_x + v = \lambda\, v, \quad s \in (0,1), \\ v(0) = w(P_0), \quad v(1) = w(P_1). \end{cases} \tag{3.28}$$

One of these pairs is given by $(w, v) = (0, \gamma_j)$ and this is the only solution which has $w \equiv 0$. If (w, v) is a solution with $w \neq 0$, then necessarily $w = \sum_{h=0}^{k-1} c_h \phi_{i+h}$ with not all c_h equal to 0. Hence, we need to find solutions of

$$\begin{cases} -\dfrac{1}{g}\,(g v_x)_x + v = \lambda\, v, \quad s \in (0,1), \\ v(0) = \displaystyle\sum_{h=0}^{k-1} c_h \phi_{i+h}(P_0), \quad v(1) = \sum_{h=0}^{k-1} c_h \phi_{i+h}(P_1) \end{cases} \tag{3.29}$$

Recall that we have defined $\chi_i^{(a,b)}(s)$ as the unique solution of (3.5) and recall that $\chi_i^{(1,0)}(s)$ and $\chi_i^{(0,1)}(s)$ are given by (3.6). Hence, if we also denote by $\xi_i(s) = \phi_i(P_0)\chi_i^{(1,0)}(s) + \phi_i(P_1)\chi_i^{(1,0)}(s)$, so that $\xi_i(0) = \phi_i(P_0)$, $\xi_i(1) = \phi_i(P_1)$ and $\tilde{v}(s) = v(s) - \sum_{h=0}^{k-1} c_h \xi_h(s)$, then $\tilde{v}$ satisfies

$$\begin{cases} -\dfrac{1}{g}\,(g \tilde{v}_x)_x + \tilde{v} = \lambda\, \tilde{v} + (\lambda - 1) \displaystyle\sum_{h=0}^{k-1} c_h \xi_h(s), \quad s \in (0,1), \\ \tilde{v}(0) = 0, \quad v(1) = 0 \end{cases} \tag{3.30}$$

By the Fredholm alternative, since $\lambda = \tau_j$, this problem has a solution if and only if $\sum_{h=0}^{k-1} c_h \xi_h(s) \perp \gamma_j$, that is

$$\sum_{h=0}^{k-1} c_h \int_0^1 g(s)\xi_h(s)\gamma_j(s)ds = 0.$$

This is equivalent to

$$\sum_{h=0}^{k-1} c_h [\phi_{i+h}(P_0) \int_0^1 g(s)\chi_i^{(1,0)}(s)\gamma_j(s)ds + \phi_{i+h}(P_1) \int_0^1 s g(s)\chi_i^{(0,1)}(s)\gamma_j(s)ds] = 0$$

and elementary integration shows that

$$\int_0^1 g(s)\chi_i^{(1,0)}(s)\gamma_j(s)ds = -\frac{1}{\lambda - 1} g(1)\gamma_j'(1) \neq 0, \quad \int_0^1 g(s)\chi_i^{(0,1)}(s)\gamma_j(s)ds = \frac{1}{\lambda - 1} g(0)\gamma_j'(0) \neq 0$$

Therefore, if we define the numbers

$$\alpha_h = \phi_{i+h}(P_0)\int_0^1 g(s)\chi_i^{(1,0)}(s)\gamma_j(s)ds + \phi_{i+h}(P_1)\int_0^1 sg(s)\chi_i^{(0,1)}(s)\gamma_j(s)ds$$

the condition above can be read as

$$\sum_{h=0}^{k-1} c_h\alpha_h = 0.$$

If $\alpha_h = 0$ for all $h = 0, 1, \dots, k-1$, this last condition is void, and for all coefficients $(c_0, c_1, \dots, c_{k-1}) \in \mathbb{R}^k$ we have a solution $\tilde{v}$ of equation (3.30) and with $v = \tilde{v} + \sum_{h=0}^{k-1} c_h\xi_h(s)$ we get an eigenfunction associated to λ. If we also consider the eigenfunction given by $(0, \gamma_j)$ then this implies that $\dim(N(A_0 - \lambda)) = k + 1$.

Also, if there exists at least an $\alpha_h \neq 0$, then the condition above represents a restriction. The coefficients $c_0, \dots, c_{k-1}$ for which we have a solution $\tilde{v}$ is now a $(k-1)$ dimensional subspace and $\dim(N(A_0 - \lambda)) = k$.

Remark 3.9. *Consider for instance the case where $\Omega = \Omega_L \cup \Omega_R$, $\Omega_L \cap \Omega_R = \emptyset$, assuming the eigenvalues of the operator $-\Delta + I$ with Neumann boundary condition are given by $\mu_1 = 1 = \mu_2 < \mu_3 < \mu_4 \leqslant \dots,$. Assume also that the eigenfunction ϕ_3 is concentrated on Ω_L and $\phi_3(P_0) \neq 0$ and that $\tau_1 = \mu_3$. In this case we have that the eigenvalue $\lambda = \tau_1 = \mu_3$ satisfies $m_g(\lambda) = 1$ and $m_a(\lambda) = 2$.*

4. Global attractors for (1.3)

Problem (1.3) can be written as an abstract semilinear evolution equation of the form

$$\begin{cases} \dot{u} = A_0 u + f_0(u) \\ u(0) = u_0 \in U_0^p \end{cases} \tag{4.1}$$

where u lives in the Banach space U_0^p and $A_0 : D(A_0) \subset U_0^p \to U_0^p$ is the linear operator defined by (3.1), with $W, V \equiv 0$.

Assume that the nonlinearity $f_0(u)(x) = f(x, u(x))$ where $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function which satisfies the following growth condition,

$$|f(x,s) - f(x,r)| \leqslant c|s-r|(1 + |s|^{\rho-1} + |r|^{\rho-1}). \tag{4.2}$$

For this ρ we first determine the space U_0^p for which we can apply the local existence result of Subsection 2.2. For this, we will chose $Y = U_0^q$, with $p \geqslant q$ and since the map f needs to transform U_0^p to U_0^q, we have that $\rho = p/q$. Moreover, we will need to have at least $p, q \geqslant N/2$.

Hence, with the notation of Section 2, we will have, using Proposition 3.1 that $\beta = 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p})$ and $\alpha = 1 - \frac{N}{2p}$. In order to obtain solutions we need $1 \leqslant \rho < \frac{\beta}{1-\alpha}$. The fact that we need to have $\frac{\beta}{1-\alpha} > 1$ imposes some restrictions on the sizes of p and q. As a matter of fact, we will need $p \geqslant q > N$ for this last restriction to hold. Moreover, if $p > N$ and $q = \frac{p(2N+1)}{2p+1} > N$, then, we have

$$\|f_0(u) - f_0(v)\|_{U_0^q} \leqslant c\|u - v\|_{U_0^p}(1 + \|u\|_{U_0^p}^{\rho-1} + \|v\|_{U_0^p}^{\rho-1}).$$

with $\rho = \frac{p}{q} = \frac{1-\frac{N}{2q}-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}{\frac{N}{2p}}$.

It follows from the results in Section 2.2 that (4.1) has a unique local solution for each initial condition $u_0 \in U_0^p$. Moreover the time of existence of solutions is bounded uniformly on bounded sets of initial data in U_0^p.

If the nonlinearity satisfies the dissipativity condition (1.4) we can easily show that the solutions are globally defined in time. This implies that we have a well defined nonlinear semigroup $T(t) : U_0^p \to U_0^p$, for $t > 0$. Moreover, if $(w_0, v_0) \in U_0^p$ and we denote by $T(t)(w_0, v_0) = (w(t), v(t))$ then w is the solution of

$$\begin{cases} w_t = \Delta w - w + f(x,w) & x \in \Omega, \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ w(0) = w_0 \end{cases} \tag{4.3}$$

and v is the solution of

$$\begin{cases} v_t = \frac{1}{g}(g v_x)_x - v + f(x,v) & x \in (0,1), \\ v(0,t) = w(P_0,t), \quad v(1,t) = w(P_1,t) \\ v(0) = v_0. \end{cases} \tag{4.4}$$

But standard regularity theory shows that $(w(t), v(t)) \in C^{1,\eta}(\bar{\Omega}) \oplus C^{1,\eta}(0,1)$ for some $\eta > 0$ and for all $t > 0$. Moreover, with the dissipativity condition (1.4) we have that if $(w_0, v_0) \in B_0$, a bounded set of U_0^p then (w, v) lies in a bounded set of $C^{1,\eta}(\bar{\Omega}) \oplus C^{1,\eta}(0,1)$, for all $t \geqslant 1$. In particular, the semigroup $T(t)$ is compact for $t > 0$ and the orbit (for all time $t \geqslant 1$) of all bounded set $B_0 \subset U_0^p$ is bounded in U_0^p. Moreover, condition (1.4) and comparison principles will imply that there exists a $M_0 > 0$ such that for any initial data u_0, there exists a time t_0 such that $\|T(t_0)(u_0)\|_{L^\infty(\Omega)\oplus L^\infty(0,1)} \leqslant M_0$ and in particular $\|T(t_0)(u_0)\|_{U_0^p} \leqslant M$ for some M. This implies that the semigroup has a global attractor $\mathcal{A}$ and that $\mathcal{A} \subset C^{1,\eta}(\bar{\Omega}) \oplus C^{1,\eta}([0,1])$.

Moreover, if we consider the system above defined in the space $C(\bar{\Omega}) \oplus L^p(0,1)$, where the operator A_0 generates a nice strongly continuous analytic semigroup and for which we have the standard theory developed in [10], we also have a well defined nonlinear semigroup which posses $\mathcal{A}$ as its attractor. In particular, any set B_0 bounded in U_0^p is attracted by $\mathcal{A}$ in the topology of $C(\bar{\Omega}) \oplus C([0,1])$.

Let us characterize the attractor $\mathcal{A}$. Since the operator A_0 is not self-adjoint we will not be able to prove that there is a Liapunov function for (1.3). However, since this problem arises as a limiting problem for a scalar parabolic equation in a bounded smooth domain we expect that its attractor is characterized by the unstable manifold of the set of equilibria.

The proof of this fact can be done, in case the set of equilibria is finite, in the following manner. We know that any solution $(w(t), v(t))$, $t \in \mathbb{R}$ in the attractor must satisfy that $w(t) \overset{t\to\pm\infty}{\longrightarrow} w_\pm^*$ where $w_\pm^*$ are solutions of the elliptic problem $-\Delta w + w = f(x,w)$ in Ω, with Neumann boundary condition. This follows from the well known fact that the system in Ω is decoupled from the system in R_0 and the fact that the system in Ω is gradient. It follows that $v(t)$ is a solution of

$$\begin{cases} v_t - \dfrac{1}{g}\left(g\, v_s\right)_s + v = f(s,v), \ \ s \in (0,1), \\ v(0) = w(t,P_0), \ \ v(1) = w(t,P_1), \end{cases}$$

which, after the change of variables $z(t) = v(t) - \xi(t,s)$ becomes,

$$\begin{cases} z_t - \dfrac{1}{g}\left(g\, z_s\right)_s + z = f(s, z+\xi(t,s)) - \xi_t(t,s) - \xi(t,s), \ \ s \in (0,1), \\ z(0) = 0, \ \ z(1) = 0, \end{cases}$$

where $\xi(t,s)$(or $\xi_{\pm}$) is the solution of

$$\begin{cases} -\dfrac{1}{g}\left(g\, \xi_s\right)_s = 0, \quad s \in (0,1) \\ \quad \xi(t,0) = w(t,P_0)(\text{or } w^*_{\pm}(P_0)), \ \ \xi(t,1) = w(t,P_1)(\text{or } w^*_{\pm}(P_1)), \end{cases} \tag{4.5}$$

Observe that ξ can be explicitely calculated. As a matter of fact, we have

$$\xi(t,s) = w(t,P_0)\chi^{(1,0)}(s) + w(t,P_1)\chi^{(0,1)}(s) \tag{4.6}$$

where functions $\chi^{(1,0)}(s)$ and $\chi^{(0,1)}$ are given by (3.6). In particular, we have that

$$\xi(t,s) \overset{t\to\pm\infty}{\longrightarrow} \xi_{\pm}(s), \ \text{uniformly in } s \in [0,1]$$

$$\xi_t(t,s) \overset{t\to\pm\infty}{\longrightarrow} 0, \ \text{uniformly in } s \in [0,1]$$

From the results in [6, 5] we have that the α and ω limit set of any point in the attractor lies in the set of equilibria. In particular, the attractor of (1.3) is described as the union of unstable manifolds of equilibria.

The following result, summarizes the results obtained in this Section,

Proposition 4.1. *Let $p > N$ and let the nonlinearity f satisfy the growth restriction* (4.2) *with $1 \leqslant \rho < \frac{2p+1}{2N+1}$ and also the dissipative condition* (1.4)*. Then, problem* (1.3) *defines a nonlinear semigroup in U^p_0, continuous for $t > 0$ and which has an attractor $\mathcal{A}$. Moreover, $\mathcal{A} \subset C^{1,\eta}(\Omega) \oplus C^{1,\eta}(0,1)$ and $\mathcal{A}$ attracts bounded sets of U^p_0 in the topology of $C(\Omega) \oplus C(0,1)$. Moreover, if the set of equilibria is finite (this is the case if all the equilibria are hyperbolic), the α and ω limit set of any point in the attractor lies in the set of equilibria. In particular, $\mathcal{A} = \bigcup_{i=1}^{n} W^u(\phi_i)$, where ϕ_i, $i = 1, \ldots, n$ are the equilibria of the system and $W^u(\phi)$ is the unstable manifold of ϕ.*

5. Some comments and remarks

In this section we make some important comments on the dynamics of problem (1.3).

5.1. **On the saddle point property.** We note that, even though the nonlinear semigroup associated to the problem (1.3) is singular at zero, it regularizes immediately. Consequently its asymptotic properties can be studied in spaces for which the problem (1.3) is well posed. From this we infer that facts like hyperbolicity of equilibria and existence of local stable and unstable manifolds as graphs can be treated as usual. For example, the hyperbolicity and dimension of the local unstable manifold of an equilibrium solution (w^*, v^*) of (1.3) is determined by the eigenvalues and generalized eigenspace of the linearization around it; that is, by the eigenvalue problem

$$\begin{cases} -\Delta w + w = \partial_u f(x, w^*) w + \mu\, w, \quad x \in \Omega \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega \\ -\dfrac{1}{g}\,(g v_x)_x + v = \partial_u f(x, v^*) v + \tau\, v, \quad s \in (0,1), \\ v(0) = 0, \quad v(1) = 0. \end{cases}$$

5.2. **Patterns.** Consider the case where the open set Ω is the union of two disjoint domains Ω^L and Ω^R, that is $\Omega = \Omega^L \cup \Omega^R$ with $\bar{\Omega}^L \cap \bar{\Omega}^R = \emptyset$ and recall that the coordinate w in (1.3) is not influenced by the coordinate v. Also, if f is independent of the spatial variable, any constant $c \in \mathbb{R}$ which satisfies $-c + f(c) = 0$ is an equilibrium solution to

$$\begin{cases} w_t = \Delta w - w + f(w) & x \in \Omega,\ t > 0 \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

More generally, if c_1 and c_2 are roots of the equation $-c+f(c) = 0$, then $w^* = c_1\chi_{\Omega^L} + c_2\chi_{\Omega^R}$ is also an equilibrium solution to (5.1). We note that, this equilibrium solution to (5.1) is asymptotically stable if and only if $f'(c_i) < 1$, for $i = 1, 2$.

Under this condtion, (w^*, v^*) will be a stable equilibrium solution to the above problem if and only if the first eigenvalue of

$$\begin{cases} -\dfrac{1}{g}\,(g z_x)_x + (1 - f'(v^*) z = \tau_{z^*} z, \quad x \in (0,1) \\ v(0) = 0, \quad v(1) = 0. \end{cases}$$

is positive. We remark that, to fulfill such condition one can simply make the interval where is posed the equation above, that is $R_0 = (0,1)$, smaller.

In this way we have been able to obtain equilibria which are linearly asymptotically stable. We can apply now the results of [3], see Theorem 2.3 and Theorem 2.5, obtaining an equilibrium solution in the dumbbell domain which is asymptotically stable.

5.3. **Contribution to the attractor $\mathcal{A}$ from the line segment.** If w^* is an equilibrium solution for (5.1), taking the pair $u_0 = (w^*, v_0)$, $v_0 \in L^p_g(0,1)$, as initial condition to (1.3) we will have a solution $(w^*, v(t, u_0))$ where v satisfies

$$\begin{cases} v_t = \dfrac{1}{g}\,(g v_x)_x - v + f(x, v), \quad x \in (0,1) \\ v(0) = c_0, \quad v(1) = c_1. \end{cases}$$

where $c_0 = w^*(P_0)$ and $c_1 = w^*(P_1)$. If $\xi = c_0\chi^{(1,0)}(s) + c_1\chi^{(0,1)}(s)$, that is, ξ is the solution of

$$\begin{cases} \frac{1}{g}\,(g\,\xi')' = 0, \quad s \in (0,1) \\ \xi(0) = c_0, \quad \xi(1) = c_1. \end{cases} \tag{5.2}$$

and $z = v - \xi$ we have that

$$\begin{cases} z_t = \frac{1}{g}\,(gz_x)_x - z - \xi + f(x, z+\xi), \quad x \in (0,1) \\ v(0) = 0, \quad v(1) = 0. \end{cases}$$

The above problem has a global attractor $\mathcal{A}_{w^*}$ in $L^p_g(0,1)$ and consequently the attractor $\mathcal{A}$ for (1.3) contains $(w^*, \mathcal{A}_{w^*} + \xi)$. In particular, if $f(0) = 0$ we can manipulate the size of the channel in order to make that the attractor $\mathcal{A}$ contains a contribution from the channel as complicated as any attractor coming from the Chafee-Infante Problem (see [7, 11]). This proves that the dinamics of (1.3) may have a very large contribution from the segment, even if the dynamics of (5.1) is trivial.

More generally, if $w(t)$, $t \in \mathbb{R}$ is a solution in the attractor $\mathcal{A}_\Omega$ for (5.1) and $\{\mathcal{A}_w(t)\}$ is the nonautonomous attractor, see [8], of the asymptotically autonomous problem

$$\begin{cases} z_t = \frac{1}{g}\,(gz_s)_s - z - \xi(t,s) - \xi_t(t,s) + f(s, z + \xi(t,s)), \quad s \in (0,1) \\ v(0) = 0, \quad v(1) = 0. \end{cases}$$

where $\xi(t,s)$ is defined as the solution of (4.5) (see also (4.6) and (3.6)) then

$$\bigcup_{t\in\mathbb{R}} \big(w(t), \xi(t,\cdot) + \mathcal{A}_w(t)\big) \subset \mathcal{A}.$$

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