PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2010-01

Continuity of dynamical structures for non-autonomous evolution equations under singular perturbations

J. Arrieta, A. N. Carvalho, J. Langa and A. Rodríguez-Bernal .

Octubre-2009

http://www.mat.ucm.es/deptos/ma
e-mail:matemática_aplicada@mat.ucm.es

CONTINUITY OF DYNAMICAL STRUCTURES FOR NON-AUTONOMOUS EVOLUTION EQUATIONS UNDER SINGULAR PERTURBATIONS

JOSÉ M. ARRIETA[1], ALEXANDRE N. CARVALHO[2], JOSÉ A. LANGA[3],
AND ANÍBAL RODRIGUEZ-BERNAL[4]

1. Introduction

Attractors for infinite dimensional dynamical systems have been the subject of the work of many mathematicians and applied scientists in the last four decades. In the case of infinite dimensional autonomous dynamical systems (nonlinear semigroups) the theory for existence of attractors and their upper semicontinuity relatively to perturbations is quite well developed. Upper semicontinuity can be described by saying that any neighborhood of the limit attractor (that is, the attractor of the limit problem) contain most of the attractors of the approximating problems. Or in other words, sequences of points in the approximating attractors converge to points in the limit one. This property allows the limit attractor to be "much larger" than the approximating ones.

To obtain that approximating and limit attractors "look the same" one needs to show lower semicontinity of these sets. In such a case all the global dynamics of approximating and limit problems are alike. However, the study of lower semicontinuity of attractors under perturbation is connected to the characterization of attractors. This characterization remains restricted to the class of autonomous dynamical systems that are gradient, asymptotically compact, with bounded set of equilibria. In such a case, the attractor is then the union of the unstable manifolds of equilibria, see [24]. Thus, results on lower semicontinuity of attractors remains resticted to such type of problems.

At this respect, it is our belief that attractors are in general the union of the unstable manifolds of normally hyperbolic invariant manifolds (see [21]). However at present time it is fair to state that lower semicontinuity is a much more demanding property than upper semicontinuity.

The study of the lower semicontinuity of attractors for autonomous semilinear differential equations in Banach spaces has its origin in the work of [26] where an abstract result has been proved and applications to partial differential equations have been considered. The results in that paper that have been used and simplified since then says that: *If the limiting equation is gradient, has a finite number n of equilibria, all of them hyperbolic, the perturbed nonlinear*

[1] Partially supported by grants MTM2006–08262, MEC; GR58/08 Grupo 920894, UCM-BSCH; PHB2006-0003-PC, MEC and SIMUMAT, Comunidad de Madrid, Spain.

[2]Partially supported by CNPq grant # 305447/2005-0 and by FAPESP grant # 03/10042-0, Brazil.

[3]Partially supported by Projects PBH2006-0003-PC, MTM2008-00088 and HF2008-0039.

[4] Partially supported by grants MTM2006–08262, MEC; GR58/08 Grupo 920894, UCM-BSCH and PHB2006-0003-PC, MEC, Spain.

semigroups vary continuously, the sets of equilibria have fixed finite cardinality and vary continuously with the parameter, and the local unstable manifolds of the perturbed problems are lower semicontinuous, then the family of attractors behaves lower semicontinuously. The proof of this result relies on results on the continuity of the equilibria and of their local unstable manifolds under perturbation. Once these local properties are proved, the lower semicontinuity of the global unstable manifolds and, consequently, the lower semicontinuity of attractors, is obtained in the following way: given a point y_0 in the limiting attractor, it lies in the unstable manifold of some equilibria. Then, we follow some solution through it, backwards in time, until it enters the neighborhood of an equilibrium point, where we have the lower semicontinuity of local unstable manifolds; we then approximate it by a point in the unstable manifold of a hyperbolic equilibria of the perturbed problem and follow the solution starting at this approximation point, now forwards in time, obtaining the approximation of y_0 by points in the perturbed attractor.

In the applications [26] considers situations for which the set of equilibria does not depend upon the parameter. That makes the application of the abstract result somewhat simpler, though the change of type in the equation makes it complicated anyway. Later in [1, 3, 6, 11, 19] the authors study different perturbation problems and consider situations for which the set of equilibria changes with the parameter while maintaining the gradient structure of the limit problem.

In all these works it becomes clear that one must account for the uniform exponential rates of the stable and unstable components of the linearization around each hyperbolic equilibrium (accomplished because the linear operator is sectorial in all these papers). This exponential rates of the stable and unstable components are usually denoted "exponential dichotomies" around the hyperbolic equilibria. In [8] the authors consider the situation for which the linear operator for the perturbed problem is sectorial whereas in the limit it is only the generator of a strongly continuous semigroup (here the uniform exponential dichotomy for the linearizations around equilibria becomes a major problem).

In all of the works cited above the perturbation is singular, in the sense that it affects the highest order terms of the equation, and the continuity of the resolvent operators of the associated linear unbounded operator is used in an essential way. When the perturbation affects lower order terms (including nonlinear ones) in the equation, we say the perturbation is regular. This case is typically a little simpler from the technical point of view.

In the realm on nonautonomous problems, there have been some attempts to approach these sort of problems. For example in [33] and [18] the authors considered the case of a regular nonautonomous perturbation of a gradient system. Also, more general problems with regular perturbations have been analyzed in [14] and later in [16, 17].

Notice that in most of the papers cited above, the limit problem is an autonomous one. Also, these papers consider regular perturbations and the type of results proved concern the proximity of the pullback attractors of the nonautonomous perturbation, to the attractor of the autonomous limit.

In this paper we consider more general situations. In particular we analyze the continuity of some dynamical structures (which belong to the attractors of the problems) under singular perturbations.

Observe that our limit problem (that is, the one subject to perturbations) is a nonautonomous one. In order to apply the general results to wide classes of problems, we allow the underlying linear equations (that is, disregarding nonlinear terms) to have some singular behavior at the intial time. Also for the sake of generality, we set the nonlinear equations using some suitable spaces $\mathcal{Y} \subset \mathcal{Z}$, which have to be chosen properly in applications. See Section 2 for some introductory material, definitions and a technical outline of the main results of the paper.

In Section 3 we study the local existece of solutions of the nonlinear equations within this general setting. Then we give sufficient conditions on the nonautonomous perturbations to prove that the perturbed processes defined by the equations, converge to the unperturbed one. This applies to both linear and nonlinear equations; see Section 4.

Then we show that in a neigborhood of a global, bounded and hyperbolic solution of the limit problem, there exists a unique global, bounded and hyperbolic solution of the approximating ones. Moreover the latter ones converge unformly in $t \in \mathbb{R}$ to the former one; see Theorem 6.1.

Note that here hyperbolicity is stated in terms of suitable exponential dichotomies of the linearizations around the nonconstant (in time) global bounded solutions. Therefore these linearizations are linear nonautonomous problems as well. These global, bounded hyperbolic solutions play the role in non autonomous problems of the hyperbolic equilibria in autonomous ones. Thus they are the natural generalizations of equilibria and can be losely denoted "nonautonomous equilibria".

The results we just described above, and many of the remaining ones in the rest of the paper, require some technical and difficult results on the roughness of expoential dichotomies under singular perturbations; see Section 5. To the best of our knowledge there were no results in this direction exepct for the ones in [27] for regular perturbations. Note that one of the main results we give here, concerns the continuity of the time dependent projection families associated to the exponential dichotomies. These time dependent projection families play the role here of the projections on the stable and unstable components in the autonomous case.

Then, in Section 7, we prove that for each of these "nonautonomous equilibria" the local unstable manifold can be described, for each $t \in \mathbb{R}$ as the graph of a suitable Lipschitz function. A detailed analysis of the dependence of these functions on the parameter of the singular perturbation, allows us to prove that the local unstable manifolds of "nonautonomous equilibria" are dynamical structures that behaves continuosly when passing to the limit.

Then we turn our attention into some global dynamical structures in Section 8. There we first prove that the families of pullback attractors are uppersemiconinuous. Then we prove that, if the limiting pullback attractor is the closure of a countable union of unstable manifolds of global, bounded hyperbolic solutions, then the pullback attractors behave continuously under singular perturbation.

Section 9 is devoted to some immediate generalizations and extensions of the previous results that require minor changes in the proofs. In Section 10 we consider some general examples, including the case when the limit problem is an autonomous gradient system. In such a case, as a consequence of all of the previous results and the results in [18, 15], we prove that if the limiting semigroup is gradient then the perturbed non-autonomous attractor is exactly the union of unstable manifolds of global hyperbolic solutions, and that in fact the pullback attractor is also a forwards time dependent exponential attractor (see Section 10.2). We remark that, detecting when a pullback attractor is also a forwards attractor has been one of the most interesting problems in the recent theory of non-autonomous dynamical systems. In general, a pullback attractor is not expected to be a forwards attractor (see [18, 34, 15]). Moreover, due to the fact that local unstable manifolds of global hyperbolic solutions are exponentially attracting, our time dependent global attractor will also attracts, uniformly on bounded sets, at an exponential rate (see [15]). Actually, what we get, now in a non-autonomous framework, is what Babin and Vishik [7] defined as a regular attractor.

Finally in Section 11 we apply the general results in this paper to some concrete problems in ordinary and partial differential equations.

2. **Notations and outline of the results**

In this section we introduce some terminology and state the main results. We start with the definition of evolution processes, which includes the definition of semigroups and processes (linear or not). Throughout the text $\mathcal{C}(\mathcal{Y})$ will denote the space of continuous, possibly nonlinear, operators defined in a Banach space $\mathcal{Y}$ and by $\mathcal{L}(\mathcal{Y})$ the space of linear and bounded operators on $\mathcal{Y}$.

Definition 2.1. *A family $\{S(t,\tau):\ t\geqslant\tau\in\mathbb{R}\}\subset\mathcal{C}(\mathcal{Y})$ satisfying*

$$\begin{aligned}&1)\ S(\tau,\tau)=I,\\&2)\ S(t,\sigma)S(\sigma,\tau)=S(t,\tau),\ \text{for each } t\geqslant\sigma\geqslant\tau,\\&3)\ (t,\tau)\mapsto S(t,\tau)y \text{ is continuous for } t>\tau,\ y\in\mathcal{Y}\end{aligned}$$

is called an evolution process. In the particular case when each $S(t,\tau)\in\mathcal{L}(\mathcal{Y})$, $t\geqslant\tau$, we say that $\{S(t,\tau):\ t\geqslant\tau\in\mathbb{R}\}$ is a linear evolution process.

An evolution process for which $S(t,\tau)=S(t+s,\tau+s)$ for all $t\geqslant\tau\in\mathbb{R}$ and for all $s\in\mathbb{R}$ is called autonomous. If we write $S(t,\tau)=S(t-\tau,0)=:U(t-\tau)$ then $\{U(t):t\geqslant 0\}$ is semigroup (linear or not). Hence, a semigroup is a family $\{S(t):t\geqslant 0\}$ such that $S(0)=I$, $S(t+s)=S(t)S(s)$, for all $t,s\geqslant 0$ and $(0,\infty)\times\mathcal{Y}\ni(t,x)\mapsto S(t)x\in\mathcal{Y}$ is a continuous.

Conversely if $\{U(t):t\geqslant 0\}$ is a semigroup and we define $S(t,\tau)=U(t-\tau)$ for all $t\geq\tau\in\mathbb{R}$, then $\{S(t,\tau):t\geqslant\tau\in\mathbb{R}\}$ is an evolution process.

Remark 2.2. *We observe that the continuity of the process $\{S(t,\tau):t\geq\tau\in\mathbb{R}\}$ at $t=\tau$ is not assumed. In fact, there are many applications for which this continuity does not hold and it is shown that it does not play any role in the study of the asymptotic behavior as long as the singularity at $t=\tau$ is integrable.*

The concept of "invariance" for evolution processes is as follows: a family $\{C(t)\}_{t\in\mathbb{R}}$ of subsets of $\mathcal{Y}$ is *positively invariant* for the process S if $S(t,\tau)C(\tau)\subset C(t)$ for each $t,\tau\in\mathbb{R}$, $t\geqslant\tau$ and it is *invariant* for the process S if $S(t,\tau)C(\tau)=C(t)$ for each $t,\tau\in\mathbb{R}$, $t\geqslant\tau$.

Important objects for the dynamics of semigroups or processes are the "globally defined solutions" that we now define

Definition 2.3.
i) Given $y_0 \in \mathcal{Y}$ *the curve* $y(t,s,y_0) = S(t,s)y_0$, *for* $t > s$, *is called "solution" of the process.*
ii) A continuous curve $\phi : \mathbb{R} \to \mathcal{Y}$ *is a* complete solution *or* global solution *of the process* $\{S(t,\tau) :\ t \geqslant \tau\}$ *if for each* $t \geqslant s$ *we have* $\phi(t) = S(t,s)\phi(s)$. *Clearly, if* $\phi : \mathbb{R} \to \mathcal{Y}$ *is a global solution and* $C(t) = \{\phi(t)\}$, *then the family* $\{C(t)\}_{t\in\mathbb{R}}$ *is invariant.*

Remark 2.4. *We observe that, given* $(t_0, y_0) \in \mathbb{R} \times \mathcal{Y}$ *there is a unique solution* $\xi : [t_0, \infty)$ *for the process* $\{S(t,\tau) : t \geqslant \tau\}$ *such that* $\xi(t_0) = y_0$. *However, the existence of a global solution* $\xi : \mathbb{R} \to \mathcal{Y}$ *which satisfy* $\xi(t_0) = y_0$ *is, in general, a much more restrictive condition on the data* y_0 *and, when a global solution exists, it may not be unique.*

We can now define the concept of unstable manifold of a global solution, which will play an essential role in what follows.

Definition 2.5. *Given* $\xi^* : \mathbb{R} \to \mathcal{Y}$, *a global solution of the process* $\{S(t,\tau) :\ t \geqslant \tau\}$, *the unstable manifold* $W^u(\xi^*)$ *of* ξ^* *is defined as* $W^u(\xi^*)=\{(t,y) \in \mathbb{R} \times \mathcal{Y}$*: there is a global solution* $\phi : \mathbb{R} \to \mathcal{Y}$ *of* $\{S(t,\tau) :\ t \geqslant \tau\}$ *such that* $\phi(t) = y$ *and* $\|\xi^*(s) - \phi(s)\|_{\mathcal{Y}} \overset{s\to-\infty}{\longrightarrow} 0\}$.

Observe that the unstable manifold is a global object in the dynamics of the process S and therefore its structure and behavior under different perturbations is not easy to describe. Hence, it seems natural to define the local version of the unstable manifolds,

Definition 2.6. *Given a global solution* $\xi^* : \mathbb{R} \to \mathcal{Y}$ *for* $\{S(t,\tau) : t \geqslant \tau\}$, *its* $\rho-$*local unstable manifold is defined as* $W^u_{loc}(\xi^*, \rho)=\{(t,y) \in \mathbb{R} \times \mathcal{Y}$*:* $\|y - \xi^*(t)\|_{\mathcal{Y}} < \rho$, *there is a global solution* ϕ *of* $\{S(t,\tau) : t \geqslant \tau\}$, *with* $\phi(t) = y$, $\|\phi(s) - \xi^*(s)\|_{\mathcal{Y}} < \rho$ *for all* $s \leqslant t$, *and* $\|\phi(s) - \xi^*(s)\|_{\mathcal{Y}} \overset{s\to-\infty}{\longrightarrow} 0\}$.

Observe that, even in the case of autonomous processes, in general it is not true that $W^u_{loc}(\xi^*, \rho) = \{(t,y) \in W^u(\xi^*) : \|y - \xi^*(t)\|_{\mathcal{Y}} < \rho\}$, being enough to consider the case when the equilibria has a homoclinic orbit (see [25, Page 298]).

Remark 2.7. *i) For the sake of notation, we will denote by* $W^u(\xi^*)(t)$ *the section at time* t *of* $W^u(\xi^*)$, *that is* $W^u(\xi^*)(t) = \{y \in \mathcal{Y},\ (t,y) \in W^u(\xi^*)\}$.

Note that $W^u(\xi^*) = \{W^u(\xi^*)(t)\}_{t\in\mathbb{R}}$ *is invariant. Similarly, we will denote by* $W^u_{loc}(\xi^*, \rho)(t)$ *the section at time* t *of* $W^u_{loc}(\xi^*, \rho)$.
ii) Observe also that from the definitions above, the section at time t *of the global unstable manifold, that is* $W^u(\xi^*)(t)$, *is obtained as the evolution at time* t *of all the sections of the local unstable manifold* $W^u_{loc}(\xi^*, \rho)(s)$ *at times* s *very far away back in time. That is, for each* $s_0 \leqslant t$, *we have*

$$W^u(\xi^*)(t) = \bigcup_{s\leqslant s_0} S(t,s)(W^u_{loc}(\xi^*, \rho)(s)) \tag{2.1}$$

Remark 2.8. *Among the class of global solutions an special role is played by those which are hyperbolic and more specifically for the subclass of "hyperbolic bounded global solutions", see below for precise definitions. In certain sense the concept of "hyperbolic bounded global solution" is, as we will see, the natural analogous for the non-autonomous case of hyperbolic equilibrium.*

As we will make precise below, a global bounded solution of a nonlinear process is called hyperbolic if a suitable linearized process has an exponential dichotomy.

Observe that for autonomous systems, even in finite dimensions, global bounded solutions are not in general hyperbolic. This is due to the invariance with respect to time translations. For example a periodic orbit or a connection between equilibria are not hyperbolic, since the time derivative of the solution is a bounded global solution of the linearized equation.

Remark 2.9. *For the applications to parabolic equations it will be natural to consider a setting with two spaces. The first one $\mathcal{Y}$ will be the phase space for the nonlinear evolution process, see* (2.4), (2.3) *and (2.7) below. The second one $\mathcal{Z}$ is the space in which some linearized process will live, see (2.8), (2.9), (2.10) and (2.11).*

Roughly speaking $\mathcal{Z}$ is a weaker space than $\mathcal{Y}$. Although in many applications this is actually the case, our general setting here does not require a strong relationship between these spaces. See Condition 2.11.

To avoid excessive repetition we will sometimes write $\mathcal{W}$ to denote indistinctly $\mathcal{Y}$ and $\mathcal{Z}$.

For non-autonomous problems, the concept of hyperbolicity is expressed in the notion of exponential dichotomy for linear evolution processes, as follows:

Definition 2.10. *Let $\mathcal{Z}$ be a Banach space and consider a linear evolution process $\{U(t,\tau) : t \geqslant \tau \in \mathbb{R}\} \subset \mathcal{L}(\mathcal{Z})$. We will say that U has exponential dichotomy with exponent ω, constant M and singularity $\gamma \in [0,1)$ if there exists a family of projections $\{Q(t) : \ t \in \mathbb{R}\} \subset \mathcal{L}(\mathcal{Z})$ such that*

i) $Q(t)U(t,s) = U(t,s)Q(s)$, for all $t \geqslant s$;
ii) The restriction of $U(t,s)$ to $R(Q(s))$ is an isomorphism from $R(Q(s))$ onto $R(Q(t))$. Its inverse is denoted by $U(s,t) : R(Q(t)) \to R(Q(s))$, for $s < t$.
iii) The following estimates hold

$$\begin{aligned} &\|U(t,s)(I - Q(s))\|_{\mathcal{L}(\mathcal{Z})} \leqslant M \max\{1, (t-s)^{-\gamma}\} e^{-\omega(t-s)}, \quad t > s \\ &\|U(t,s)Q(s)\|_{\mathcal{L}(\mathcal{Z})} \leqslant M e^{\omega(t-s)}, \quad t \leqslant s. \end{aligned} \tag{2.2}$$

We note that we do not assume that $Q(t)$ has finite rank but, of course, in such a case, by ii) in the Definition, the rank of $Q(t)$ is independent of $t \in \mathbb{R}$.

Now we make precise the type of evolution processes, as well as the perturbations and linearizations that we will consider in this paper.

We consider a family of semilinear problems in a Banach space $\mathcal{Y}$, which will be our common phase space for the different problems we are considering. First we consider a semilinear "limiting" problem

$$\begin{cases} \dot{y} = \mathfrak{B}_0 y + f(t,y), \quad t > \tau \\ y(\tau) = y_0 \end{cases} \tag{2.3}$$

and, second, a singular perturbation of it

$$\begin{cases} \dot{y} = \mathfrak{B}_\eta y + f(t,y), \quad t > \tau \\ y(\tau) = y_0 \end{cases} \tag{2.4}$$

with $\eta \in (0,1]$. To unify the notations, observe that (2.3) corresponds to $\eta = 0$ in (2.4).

Concerning the linear part of the equations, let us assume the following

Condition 2.11. *Let $\mathcal{Z}$ and $\mathcal{Y}$ be Banach spaces such that*

$$\mathcal{Y} \subset \mathcal{Z}$$

and $\mathfrak{B}_\eta$ be a linear (unbounded) operator which generates a singular semigroup $\{e^{\mathfrak{B}_\eta t} : t \geq 0\}$ in $\mathcal{L}(\mathcal{Z})$ and in $\mathcal{L}(\mathcal{Y})$ and with $e^{\mathfrak{B}_\eta t} \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ for each $t > 0$, $\eta \in [0,1]$. Assume that there are constants $\gamma \in [0,1)$, $\beta > 0$ and $M \geqslant 1$, independent of $\eta \in [0,1]$, such that

$$\begin{aligned} \|e^{\mathfrak{B}_\eta t}\|_{\mathcal{L}(\mathcal{Z})} &\leqslant M t^{-\gamma} e^{-\beta t} \\ \|e^{\mathfrak{B}_\eta t}\|_{\mathcal{L}(\mathcal{Y})} &\leqslant M t^{-\gamma} e^{-\beta t} \\ \|e^{\mathfrak{B}_\eta t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} &\leqslant M t^{-\gamma} e^{-\beta t} \end{aligned} \tag{2.5}$$

for all $\eta \in [0,1]$.

Remark 2.12. *Observe that the condition (2.5) is meant to deal with two different sources of singular behavior at $t = 0$. First, it accounts for the possibility that the semigroup is singular (in particular, not continuous at $t = 0$) in either $\mathcal{Z}$ or $\mathcal{Y}$. Second, it accounts for the regularizing estimates from $\mathcal{Z}$ to $\mathcal{Y}$. A common situation is when $\mathcal{Y}$ coincides with the fractional power space in $\mathcal{Z}$ associated to the operators $\mathfrak{B}_\eta$, see [27]. This particular case will be consider in detail later on in Section 10.1. Note that we could consider three different exponents instead of a single one γ in Condition 2.11. This different exponents would reflect different types of singularities in different spaces. However for the sake of simplicity we chose the largest of the three.*

With respect to the nonlinear term, we assume the following

Condition 2.13. *The function $f : \mathbb{R} \times \mathcal{Y} \to \mathcal{Z}$ is continuously differentiable, bounded and globally Lipschitz continuous in the second variable uniformly in the first variable. Moreover, if we denote by $D_y f(t,y) \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$ the partial derivative of f with respect to the second variable in (t,y), we have that $\|D_y f(t,y)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})} \leqslant L$ for all $y \in \mathcal{Y}$ and $t \mapsto D_y f(t,y) \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$ is locally Hölder continuous. We also asume $\sup_{t\in\mathbb{R}} \|D_y f(t,y) - D_y f(t,\tilde{y})\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})} \leqslant C(\|y - \tilde{y}\|_{\mathcal{Y}})$ where $C(s) \to 0$ as $s \to 0$.*

From this we have that, for any bounded set $B \subset \mathcal{Y}$ and for any $\delta > 0$ small there is an $\epsilon > 0$ such that

$$\|f(t,y) - f(t,y_0) - D_y f(t,y_0)(y - y_0)\|_{\mathcal{Z}} \leqslant \delta \|y - y_0\|_{\mathcal{Y}} \tag{2.6}$$

for all $y_0 \in B$, $\|y - y_0\|_{\mathcal{Y}} \leqslant \epsilon$, $t \in \mathbb{R}$.

We will show that in this setting, problems (2.3) and (2.4) are globally well posed in $\mathcal{Z}$ and in $\mathcal{Y}$ and for any $y_0 \in \mathcal{W}$ (either $\mathcal{Z}$ or $\mathcal{Y}$) we have the following representation of the solutions

$$T_\eta(t,\tau)y_0 = e^{\mathfrak{B}_\eta(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}_\eta(t-s)} f(s, T_\eta(s,\tau)y_0)\, ds, \quad \eta \in [0,1]. \tag{2.7}$$

In this way, $\{T_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ defines a nonlinear evolution process in $\mathcal{W}$, for any $0 \leqslant \eta \leqslant 1$. Moreover, if $y_0 \in \mathcal{Z}$ then the solution enters $\mathcal{Y}$ instantaneously, see Proposition 3.3.

Hence, if $\xi^*_\eta(\cdot)$ is a global solution of (2.4), $\eta \in [0,1]$, then $\xi^*_\eta : \mathbb{R} \to \mathcal{Y}$ and we consider the linearization of (2.4) around $\xi^*_\eta(\cdot)$

$$\begin{cases} \dot{y} = \mathfrak{B}_\eta y + (D_y f(t, \xi^*_\eta(t)))y \\ y(\tau) = y_0. \end{cases} \tag{2.8}$$

By technical reasons that will come out clear later, if $\xi^*_0(\cdot) : \mathbb{R} \to \mathcal{Y}$ is a global solution of (2.3), we also need to consider the auxiliar problem, for $\eta \in [0,1]$,

$$\begin{cases} \dot{y} = \mathfrak{B}_\eta y + (D_y f(t, \xi^*_0(t)))y \\ y(\tau) = y_0. \end{cases} \tag{2.9}$$

We will prove that the solutions of the linear nonautonomous problems (2.8) and (2.9) define the associated evolution processes $\{U_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ and $\{V_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ in $\mathcal{L}(\mathcal{Z})$ and in $\mathcal{L}(\mathcal{Y})$. Even more the process belong to $\mathcal{L}(\mathcal{Z},\mathcal{Y})$; see Corollary 3.6. These evolution processes have the representation

$$U_\eta(t,\tau)y_0 = e^{\mathfrak{B}_\eta(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}_\eta(t-s)}(D_y f(s,\xi^*_\eta(s)))U_\eta(s,\tau)y_0 ds \tag{2.10}$$

$$V_\eta(t,\tau)y_0 = e^{\mathfrak{B}_\eta(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}_\eta(t-s)}(D_y f(s,\xi^*_0(s)))V_\eta(s,\tau)y_0 ds \tag{2.11}$$

respectively. Note that $U_0(t,\tau) = V_0(t,\tau)$ for all $t \geqslant \tau \in \mathbb{R}$.

We can finally make precise the concept of global hyperbolic solution as follows:

Definition 2.14. *A global solution $\xi^*_\eta(\cdot) : \mathbb{R} \to \mathcal{Y}$ of* (2.4)*, for $\eta \in [0,1]$, is a "hyperbolic global solution" if the linearized evolution process $\{U_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ in (2.10), associated to* (2.8)*, has exponential dichotomy in $\mathcal{Z}$, in the sense of Definition 2.10.*

We state now in a precise manner the way in which problems (2.3) and (2.4) are close to each other. We assume the following hypothesis,

Condition 2.15. *For $0 \leqslant \gamma < 1$ as in Condition 2.11, there exists a positive function $\rho(\eta,T)$ with $\rho(\eta,T) \stackrel{\eta\to 0}{\longrightarrow} 0$, such that for all $T > 0$, we have*

$$\begin{aligned} \sup_{0<t\leqslant T} t^\gamma \|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Z})} \leqslant \rho(\eta,T), \\ \sup_{0<t\leqslant T} t^\gamma \|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant \rho(\eta,T). \end{aligned} \tag{2.12}$$

Remark 2.16. *Observe that last inequality implies, in particular,*

$$\sup_{0<t\leqslant T} t^\gamma \|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Y})} \leqslant \rho(\eta,T),$$

which will be used in what follows.

With this convergence of the linear semigroups we will prove in Section 4 the following result on convergence of the nonlinear evolution processes.

Proposition 2.17. *If Condition 2.11, Condition 2.13 and Condition 2.15 hold, then for each $T, r > 0$, the processes given by (2.7), satisfy*

$$\sup_{0<t-\tau\leqslant T} \sup_{\tau\in\mathbb{R}} \sup_{\|y\|_{\mathcal{Y}}\leqslant r} (t-\tau)^{\gamma} \|T_\eta(t,\tau)y - T_0(t,\tau)y\|_{\mathcal{Y}} \xrightarrow{\eta\to 0} 0. \tag{2.13}$$

With this convergence of the nonlinear evolution processes we can show the following two key results on this work. First we have the following result on the continuity of global bounded hyperbolic solutions, which is proved in Section 6.

Theorem 2.18. *Under the same hypotheses of Proposition 2.17, let us denote by $\xi_0^*(\cdot)$ a hyperbolic bounded global solution of (2.3). Then, $\xi_0^*(\cdot)$ is an isolated solution, that is, there exists $\epsilon_0 > 0$ such that there is no other global bounded solution of (2.3), $x(t)$, with $\|x(t) - \xi_0^*(t)\|_{\mathcal{Y}} < \epsilon_0$ for all $t \in \mathbb{R}$. Moreover, there exists $\epsilon_0 > 0$ and $\eta_0 > 0$ such that for all $0 \leqslant \eta \leqslant \eta_0$, there exists a unique global solution $\xi_\eta^*(\cdot)$ of (2.4) with*

$$\sup_{t\in\mathbb{R}} \|\xi_\eta^*(t) - \xi_0^*(t)\|_{\mathcal{Y}} \leqslant \epsilon_0.$$

This solution $\xi_\eta^*(\cdot)$ is also hyperbolic and bounded and satisfies

$$\sup_{t\in\mathbb{R}} \|\xi_\eta^*(t) - \xi_0^*(t)\|_{\mathcal{Y}} \to 0 \text{ as } \eta \to 0.$$

Then, in Section 7 we prove the following result on the continuity of local unstable manifolds of global bounded hyperbolic solutions.

Theorem 2.19. *Assume the same hypotheses of Proposition 2.17, and assume $\xi_0^*(\cdot)$ and $\xi_\eta^*(\cdot)$ are as in Theorem 2.18.*

Then there exist $\eta_0, \rho > 0$ small enough such that for all $0 \leqslant \eta \leq \eta_0$ there exists a $\rho-$local unstable manifold $W^u_{loc}(\xi_\eta^*, \rho)$ for problem (2.4) and a $\rho-$local unstable manifold $W^u_{loc}(\xi_0^*, \rho)$ for problem (2.3). Moreover, these ρ-local unstable manifolds behave continuously in $\mathcal{Y}$ as $\eta \to 0$, that is, given $\tau \in \mathbb{R}$,

$$\sup_{t\leq\tau} (\mathrm{dist}_{\mathcal{Y}}(W^u_{loc}(\xi_\eta^*, \rho)(t), W^u_{loc}(\xi_0^*, \rho)(t)) + \mathrm{dist}_{\mathcal{Y}}(W^u_{loc}(\xi_0^*, \rho)(t), W^u_{loc}(\xi_\eta^*, \rho)(t))) \to 0, \text{ as } \eta \to 0.$$

where in general $\mathrm{dist}_{\mathcal{W}}$ denotes the Hausdorff semimetric in $\mathcal{W}$ defined as

$$\mathrm{dist}_{\mathcal{W}}(A, B) = \sup_{a\in A} \inf_{b\in B} ||a - b||_{\mathcal{W}}. \tag{2.14}$$

We prove Theorem 2.19 by proving the existence of local unstable manifolds of ξ_η^* as graphs of Lipschitz functions. For this, given $y(t)$ a solution of (2.4), by the change of variable $y(t) - \xi_\eta^*(t)$, we translate all the dynamics around $\xi_\eta^*(\cdot)$ to the zero solution, and (2.4) transforms into

$$\dot{y} = (\mathfrak{B}_\eta + D_y f(t, \xi_\eta^*(t)))y + h_\eta(t, y), \quad y(\tau) = y_0 \in \mathcal{Y} \tag{2.15}$$

where $h_\eta : \mathbb{R} \times \mathcal{Y} \to \mathcal{Z}$ is given by

$$h_\eta(t, y) = f(t, y + \xi_\eta(t)) - D_y f(t, \xi_\eta(t))y,$$

is differentiable with $h_\eta(t, 0) = 0$, $D_y h_\eta(t, 0) = 0$, $\eta \in [0, \eta_0]$.

Thus we prove that there exits a suitably small neighborhood $\mathcal{N}$ of $y=0$ in $\mathcal{Y}$ such that, for all $0 \leqslant \eta \leqslant \eta_0$, there exist mappings $\mathbb{R}\times\mathcal{N} \ni (\tau, y) \mapsto \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)y)$ into $\mathcal{Y}$, such that the local unstable manifolds $W^u_{\mathrm{loc},\eta}(\xi^*_\eta)$ for ξ^*_η are given by

$$W^u_{\mathrm{loc},\eta}(\xi^*_\eta)(\tau) = \{\xi^*_\eta(\tau) + y,\ y = w + \Sigma_\eta^{*,u}(\tau, w) \in \mathcal{N}, \quad w = Q_\eta(\tau)y \in R(Q_\eta(\tau)) \cap \mathcal{N}\},$$

where $Q_\eta(\tau)$ denote the projections associated to the hyperbolicity of ξ^*_η, see Definition 2.14.

Finally, we prove the continuity of the local unstable manifolds by showing that, for any $\tau \in \mathbb{R}$,

$$\sup_{t\leqslant\tau}\sup_{y\in\mathcal{N}}\left\{\|Q_\eta(t)y - Q_0(t)y\|_{\mathcal{Y}} + \|\Sigma_\eta^{*,u}(t, Q_\eta(t)y) - \Sigma_0^{*,u}(t, Q_0(t)y)\|_{\mathcal{Y}}\right\} \xrightarrow{\eta\to 0} 0.$$

The proof of Theorems 2.18 and 2.19 rely on a careful study of the linearized equations around the solutions ξ^*_η, analyzing in detail the linear evolution process they generate, (2.10), the exponential dichotomy they have and showing the convergence of the linear processes and the projections associated to the exponential dichotomies. As a matter of fact, we are able to show the following general and important result on the behavior of the exponential dichotomies under rather general perturbations of an evolution process. Note that this result, which we prove in Section 5, is of independent interest and complements the results in [27].

Theorem 2.20. *Assume we have an evolution process $\{W_0(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ in $\mathcal{Z}$ which has an exponential dichotomy with constant M_0, exponent ω_0 and projections $\{Q_0(t) : t \in \mathbb{R}\}$. Let $\{W_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ be an evolution process such that, for each $T > 0$ and some $\gamma > 0$,*

$$L_T = \sup_{\eta\in[0,1]}\sup_{\tau\in\mathbb{R}}\sup_{0\leqslant t-\tau\leqslant T}(t-\tau)^\gamma\|W_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Z})} < \infty \tag{2.16}$$

and

$$\sup_{\tau\in\mathbb{R}}\sup_{0\leqslant t-\tau\leqslant T}\{(t-\tau)^\gamma\|W_\eta(t,\tau) - W_0(t,\tau)\|_{\mathcal{L}(\mathcal{Z})}\} \xrightarrow{\eta\to 0} 0. \tag{2.17}$$

Then, given $0 < \omega < \omega_0$ and $M_1 > M_0$, there exists $\eta_0 > 0$ such that, for all $\eta \in [0, \eta_0]$, the evolution process $\{W_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has exponential dichotomy with constant $M := KM_1$, exponent ω and projections $\{Q_\eta(t) : t \in \mathbb{R}\}$ were

$$K = \sup_{\eta\in[0,1]}\sup_{\ell\leqslant t-\tau\leqslant 2\ell}\{e^{\omega(t-\tau)}\|W_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Z})}, (t-\tau-\ell)^\gamma\|W_\eta(t,\tau+\ell)\|_{\mathcal{L}(\mathcal{Z})}e^{\omega(t-\tau)}\}$$

and $\ell > 0$ is such that $Me^{-(\omega_0-\omega)\ell} < 1$. Furthermore,

$$\sup_{t\in\mathbb{R}}\|Q_\eta(t) - Q_0(t)\|_{\mathcal{L}(\mathcal{Z})} \xrightarrow{\eta\to 0} 0.$$

Note that compared with the results in [27], Theorem 2.20 applies to processes which are singular at the initial time. In addition, the continuity of the projections is also proved.

3. Local well posedness for linear and nonlinear singular processes

In general, if for fixed η, $\mathfrak{B}_\eta$ generates a strongly (analytic or not) continuous semigroup then the nonlinear process $T_\eta(t,\tau)$ and the linear ones, $V_\eta(t,\tau)$ and $U_\eta(t,\tau)$, can be obtained in a more or less standard manner (see [27, Theorem 7.1.3] or [28]). On the other hand, if the semigroup generated by $\mathfrak{B}_\eta$ is singular at the initial time, see Condition 2.11, the processes can be obtained with a fixed point argument following [2].

In fact, let $\mathcal{W} = \mathcal{Y}$ or $\mathcal{Z}$ and consider the semilinear problem

$$\begin{cases} \dot{y} = \mathfrak{B}y + f(t, y), \\ y(\tau) = y_0 \in \mathcal{W} \end{cases} \tag{3.1}$$

where the operator $\mathfrak{B}$ generates a semigroup which satisfies Condition (2.11) and the non-linearity $f : \mathbb{R} \times \mathcal{Y} \to \mathcal{Z}$ is a locally Lipschitz and bounded map which satisfies,

$$\begin{aligned} &\|f(t, y)\|_{\mathcal{Z}} \leqslant c(1 + \|y\|_{\mathcal{Y}}^{\rho}), \\ &\|f(t, y) - f(t, \tilde{y})\|_{\mathcal{Z}} \leqslant c(1 + \|y\|_{\mathcal{Y}}^{\rho-1} + \|\tilde{y}\|_{\mathcal{Y}}^{\rho-1})\|y - \tilde{y}\|_{\mathcal{Y}}, \end{aligned} \tag{3.2}$$

where $1 \leqslant \rho < \frac{1}{\gamma}$.

Our first task is to give meaning to a solution of the problem (3.1).

Definition 3.1. *For $y_0 \in \mathcal{W}$, we will say that $y : (\tau, T) \to \mathcal{Y}$ is a solution for the initial value problem* (3.1) *if $(\tau, T) \ni t \mapsto y(t) \in \mathcal{Y}$ is continuous, $(t - \tau)^\gamma \|y(t)\|_{\mathcal{Y}}$ is bounded for $t \to \tau$ and*

$$y(t) = e^{\mathfrak{B}(t-\tau)} y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f(s, y(s)) ds. \tag{3.3}$$

Remark 3.2. *Observe that we do not require the solution to be continuous in $\mathcal{Y}$ at $t = \tau$ and in general the solution will not be continuous at $t = \tau$. This is the case, for instance, if $y_0 \in \mathcal{Z}$ but $y_0 \notin \mathcal{Y}$.*

Also, this happens if $f \equiv 0$, so $y(t, y_0) = e^{\mathfrak{B}(t-\tau)} y_0$, which is not assumed to be continuous in $\mathcal{Y}$ at $t = \tau$.

We are able to show the following result, which is obtained very much in the spirit of the results in [2].

Proposition 3.3. *In the conditions above, for each $y_0 \in \mathcal{W}$ (either $\mathcal{Z}$ or $\mathcal{Y}$) there is a unique solution $y(t, \tau, y_0) = T(t, \tau) y_0$ of* (3.1) *defined on a maximal interval of existence $(\tau, \tau_{\max}(y_0))$.*

Moreover, we have

i) If $y_0 \in \mathcal{W}$, the time of existence $\tau_{max}(y_0)$ can be chosen uniformly in bounded sets of $\mathcal{W}$, in particular the following continuation result holds:

$$\text{either } \tau_{max}(y_0) = +\infty \quad \text{ or } \quad \limsup_{t \to \tau_{\max}} \|y(t, y_0)\|_{\mathcal{Y}} = +\infty.$$

ii) The time of existence is upper semicontinuous in $\mathcal{W}$, that is, if $y_n \to y_0$ in $\mathcal{W}$ then $\liminf_{n \to \infty} \tau_{max}(y_n) \geqslant \tau_{max}(y_0)$.

iii) The solution is continuous with respect to the initial conditions in the following sense: if $y_0 \in \mathcal{W}$ and if $\tau_0 < \tau_{max}(y_0)$, then for $\delta > 0$ small we have

$$\|y(t, y_0) - y(t, y_0')\|_{\mathcal{Y}} \leqslant C(t - \tau)^{-\gamma} \|y_0 - y_0'\|_{\mathcal{W}}, \qquad t \in (\tau, \tau_0], \quad \|y_0 - y_0'\|_{\mathcal{W}} < \delta. \tag{3.4}$$

Proof: Since the linear part is singular at $t=\tau$ we search for solutions for the semilinear problem with the same kind of singularity; that is, we seek for solutions in

$$K(\tau_0,\mu)=\{y\in C((\tau,\tau_0],\mathcal{Y}):\sup_{t\in(\tau,\tau_0]}(t-\tau)^{\gamma}\|y(t)\|_{\mathcal{Y}}\leqslant\mu\},$$

with the metric

$$\|y-\bar{y}\|_{K(\tau_0,\mu)}=\sup_{t\in(\tau,\tau_0]}(t-\tau)^{\gamma}\|y(t)-\bar{y}(t)\|_{\mathcal{Y}}.$$

It is not difficult to see that, with this metric, $K(\tau_0,\mu)$ is a complete metric space.

Assume that $y_0\in\mathcal{W}$ and on $K(\tau_0,\mu)$ define the map

$$U(y)(t)=e^{\mathfrak{B}(t-\tau)}y_0+\int_{\tau}^{t}e^{\mathfrak{B}(t-s)}f(s,y(s))ds.$$

For suitably chosen $\mu>0$ and $\tau_0>\tau$, with $0<\tau_0-\tau\leq 1$ small enough, we will show that, U takes $K(\tau_0,\mu)$ into itself and it is a strict contraction, uniformly for y_0 in bounded subsets of $\mathcal{W}$. For this we will use below C to denote a constant whose value may change from line to line.

Let us fix $R>0$ and consider initial conditions $y_0\in\mathcal{W}$ with $\|y_0\|_{\mathcal{W}}\leq R$. Define $M=\sup_{\{0<s\leq 1,\|y_0\|_{\mathcal{W}}\leq R\}}s^{\gamma}\|e^{\mathfrak{B}s}y_0\|$ and consider $\mu=M+1$.

Hence,

$$\begin{aligned}
\|U(y)(t)-e^{\mathfrak{B}(t-\tau)}y_0\|_{\mathcal{Y}} &\leqslant C\int_{\tau}^{t}(t-\theta)^{-\gamma}\|f(\theta,y(\theta))\|_{\mathcal{Z}}\,d\theta\\
&\leqslant C\int_{\tau}^{t}(t-\theta)^{-\gamma}(1+\|y(\theta)\|_{\mathcal{Y}}^{\rho})\,d\theta\\
&\leqslant \frac{C}{1-\gamma}\,(t-\tau)^{1-\gamma}+C\int_{\tau}^{t}(t-\theta)^{-\gamma}(\theta-\tau)^{-\rho\gamma}((\theta-\tau)^{\gamma}\|y(\theta)\|_{\mathcal{Y}})^{\rho}\,d\theta\\
&\leqslant \frac{C}{1-\gamma}\,(t-\tau)^{1-\gamma}+C\int_{\tau}^{t}(t-\theta)^{-\gamma}\,(\theta-\tau)^{-\rho\gamma}\,d\theta\;\mu^{\rho} \qquad (3.5)\\
&\leqslant \frac{C}{1-\gamma}\,(t-\tau)^{1-\gamma}+Ck^{\rho}(t-\tau)^{1-\gamma-\rho\gamma}\int_{0}^{1}(1-\theta)^{-\gamma}\,\theta^{-\rho\gamma}\,d\theta\\
&\leqslant \frac{C}{1-\gamma}\,(t-\tau)^{1-\gamma}+C\mu^{\rho}(t-\tau)^{1-\gamma-\rho\gamma}\mathcal{B}(1-\gamma,1-\rho\gamma)\\
&=(t-\tau)^{-\gamma}(\frac{C}{1-\gamma}\,(t-\tau)+C\mu^{\rho}(t-\tau)^{1-\rho\gamma}\mathcal{B}(1-\gamma,1-\rho\gamma))\\
&=h(t)(t-\tau)^{-\gamma},
\end{aligned}$$

where $h(t)=\frac{C}{1-\gamma}\,(t-\tau)+C\mu^{\rho}(t-\tau)^{1-\rho\gamma}\mathcal{B}(1-\gamma,1-\rho\gamma)\to 0$ as $t\to\tau^{+}$, where $\mathcal{B}$ denotes the Beta function, i.e. $\mathcal{B}(a,b)=\int_0^1 r^{a-1}(1-r)^{b-1}dr$ for $a,b>0$. Note that we have used that $\rho<\frac{1}{\gamma}$. Hence, we can choose $\tau_0>\tau$, such that for all $\tau<t<\tau_0$, $h(t)<1/2$ and therefore

$$(t-\tau)^{\gamma}\|U(y)(t)\|_{\mathcal{Y}}\leq(t-\tau)^{\gamma}\|e^{\mathfrak{B}(t-\tau)}y_0\|_{\mathcal{Y}}+\frac{1}{2}\leqslant M+\frac{1}{2}<\mu$$

Hence, U takes $K(\tau_0,\mu)$ into itself.

Furthermore, for $y,\bar{y}\in K(\tau_0,\mu)$,

$$\begin{aligned}
&\|U(y)(t)-U(\bar{y})(t)\|_{\mathcal{Y}}\leqslant C\int_\tau^t (t-\theta)^{-\gamma}\|f(\theta,y(\theta))-f(\theta,\bar{y}(\theta))\|_{\mathcal{Z}}\,d\theta\\
&\leqslant C\int_\tau^t (t-\theta)^{-\gamma}(1+\|y(\theta)\|_{\mathcal{Y}}^{\rho-1}+\|\bar{y}(\theta)\|_{\mathcal{Y}}^{\rho-1})\|y(\theta)-\bar{y}(\theta)\|_{\mathcal{Y}}\,d\theta\\
&\leqslant\left(\frac{C}{1-\gamma}\,(t-\tau)^{1-\gamma}+C\int_\tau^t (t-\theta)^{-\gamma}\,(\theta-\tau)^{-\rho\gamma}\,d\theta\;\mu^{\rho-1}\right)\|y-\bar{y}\|_{K(\tau_0,\mu)} \qquad (3.6)\\
&\leqslant\left(\frac{C}{1-\gamma}(t-\tau)^{1-\gamma}+C\mu^{\rho-1}(t-\tau)^{1-\gamma-\rho\gamma}\mathcal{B}(1-\gamma,1-\rho\gamma)\right)\|y-\bar{y}\|_{K(\tau_0,\mu)}\\
&=h(t)(t-\tau)^{\gamma}\|y-\bar{y}\|_{K(\tau_0,\mu)}\leq\frac{1}{2}(t-\tau)^{\gamma}\|y-\bar{y}\|_{K(\tau_0,\mu)}
\end{aligned}$$

Hence, for $\tau<t\leqslant\tau_0$ we have

$$(t-\tau)^{\gamma}\|U(y)(t)-U(\bar{y})(t)\|_{\mathcal{Y}}\leqslant\frac{1}{2}\|y-\bar{y}\|_{K(\tau_0,\mu)}.$$

After this, we have that U takes $K(\tau_0,\mu)$ into itself and it is a contraction uniformly with respect to y_0 with $\|y_0\|_{\mathcal{W}}\leq R$. It follows from the Banach contraction principle that U has a unique fixed point in $K(\tau_0,\mu)$. Hence, the initial value problem (3.1) has a unique solution in the sense of Definition 3.1.

As for the continuity with respect to the initial condition, it follows that

$$\begin{aligned}
&\|y(t,\tau,y_0)-y(t,\tau,\bar{y}_0)-e^{\mathfrak{B}(t-\tau)}(y_0-\bar{y}_0)\|_{\mathcal{Y}}\\
&\leqslant C\int_\tau^t (t-\theta)^{-\gamma}\|f(\theta,y(\theta,\tau,y_0))-f(\theta,y(\theta,\tau,\bar{y}_0))\|_{\mathcal{Z}}\,d\theta\\
&\leqslant C\int_\tau^t (t-\theta)^{-\gamma}(1+\|y(\theta,\tau,y_0)\|_{\mathcal{Y}}^{\rho-1}+\|y(\theta,\tau,\bar{y}_0)\|_{\mathcal{Y}}^{\rho-1})\|y(\theta,\tau,y_0)-y(\theta,\tau,\bar{y}_0)\|_{\mathcal{Y}}d\theta\\
&\leqslant C\int_\tau^t (t-\theta)^{-\gamma}\|y(\theta,\tau,y_0)-y(\theta,\tau,\bar{y}_0)\|_{\mathcal{Y}}\,d\theta\\
&+C\int_\tau^t (t-\theta)^{-\gamma}(\theta-\tau)^{-\rho\gamma}2\mu^{\rho-1}(\theta-\tau)^{\gamma}\|y(\theta,\tau,y_0)-y(\theta,\tau,\bar{y}_0)\|_{\mathcal{Y}}\,d\theta
\end{aligned}$$

Now denote $E(t)=\|y(t,\tau,y_0)-y(t,\tau,\bar{y}_0)\|_{\mathcal{Y}}$ and $\zeta(\tau_0)=\sup_{t\in(\tau,\tau_0]}(t-\tau)^{\gamma}E(t)$. Then we get

$$\begin{aligned}
E(t)&\leqslant C(t-\tau)^{-\gamma}\|y_0-\bar{y}_0\|_{\mathcal{W}}\\
&\quad+\left(C\int_\tau^t (t-\theta)^{-\gamma}(\theta-\tau)^{-\gamma}\,\mathrm{d}\theta+C\int_\tau^t (t-\theta)^{-\gamma}\,(\theta-\tau)^{-\rho\gamma}\,d\theta\;k^{\rho-1}\right)\zeta(\tau_0)
\end{aligned}$$

Thus

$$E(t)\leqslant C(t-\tau)^{-\gamma}\|y_0-\bar{y}_0\|_{\mathcal{W}}+C\left((t-\tau)^{1-2\gamma}+(t-\tau)^{1-\gamma-\rho\gamma}\right)\zeta(\tau_0)$$

Hence,

$$\zeta(\tau_0) \leqslant C\|y_0 - \bar{y}_0\|_{\mathcal{W}} + C\left((\tau_0-\tau)^{1-\gamma} + (\tau_0-\tau)^{1-\rho\gamma}\right)\zeta(\tau_0).$$

Hence, choosing if necessary τ_0 closer to τ and using that $\gamma < 1$ and $\rho < 1/\gamma$, we obtain that $C\left((\tau_0-\tau)^{1-\gamma} + (\tau_0-\tau)^{1-\rho\gamma}\right) \leq 1/2$, which implies that for each $y, \bar{y} \in K(\tau_0, \mu)$,

$$\sup_{t\in(\tau,\tau_0]} (t-\tau)^{\gamma}\|y(t) - \bar{y}(t)\|_{\mathcal{Y}} \leqslant 2C\|y_0 - \bar{y}_0\|_{\mathcal{W}}.$$

Note that the constant C in the right hand aside above is uniform for $y_0, \bar{y}_0$ with $\|y_0\|_{\mathcal{W}}, \|\bar{y}_0\|_{\mathcal{W}} \leq R$. This is saying that the solutions of the semilinear problem (3.1) behave exactly as the solutions of the corresponding linear problem, also with respect to initial conditions, that is

$$\|y(t) - \bar{y}(t)\|_{\mathcal{Y}} \leqslant C(t-\tau)^{-\gamma}\|y_0 - \bar{y}_0\|_{\mathcal{W}},\ t \in (\tau, \tau_0].$$

Next we observe that the continuation of solutions holds in the following sense, if a solution defined on its maximal interval of existence $y(\cdot, y_0) : (0, \tau_{\max})$, then either $\tau_{\max} = +\infty$ or $\limsup_{t\to\tau_{\max}} \|y(t, y_0)\|_{\mathcal{Y}} = +\infty$. This is accomplished simply noting that the choice of τ_0 in the proof of existence can be made uniform in bounded subsets of $\mathcal{W}$. □

Remark 3.4. *Observe that if we have the more restrictive condition* $1 \leqslant \rho < \frac{1}{\gamma} - 1$, *then from (3.5) we get that the solution of (3.3) constructed in Proposition 3.3 satisfies*

$$\|y(t) - e^{\mathfrak{B}(t-\tau)}y_0\|_{\mathcal{Y}} \to 0, \quad \text{as } t \to \tau. \tag{3.7}$$

In particular, we get

Proposition 3.5. *If* $\mathbb{R} \ni t \mapsto D(t) \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ *continuous, the linear equation*

$$\begin{cases} \dot{y} = \mathfrak{B}y + D(t)y, \\ y(\tau) = y_0 \in \mathcal{W}. \end{cases} \tag{3.8}$$

defines a process $U(t,\tau)$ *in* $\mathcal{L}(\mathcal{W}, \mathcal{Y})$, *given by*

$$U(t,\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)}D(s)U(s,\tau)y_0 ds.$$

such that

$$\|U(t,\tau)\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} \leqslant C(t-\tau)^{-\gamma} \tag{3.9}$$

which moreover satisfies $U(t,\tau) \in \mathcal{L}(\mathcal{Z})$ *and*

$$\|U(t,\tau)\|_{\mathcal{L}(\mathcal{Z})} \leqslant C(t-\tau)^{-\gamma} \tag{3.10}$$

for $0 \leqslant t-\tau \leqslant T$ *and* $C = C(\tau, T)$.

Proof: With $f(t,u) = D(t)u$, from Proposition 3.3 we have obtained the process which satisfies (3.9). Denoting $y(t) = U(t,\tau)y_0$, we have

$$\|y(t)\|_{\mathcal{Z}} \leqslant C(t-\tau)^{-\gamma}\|y_0\|_{\mathcal{Z}} + C\int_\tau^t (t-s)^{-\gamma}\|y(s)\|_{\mathcal{Y}}\, ds.$$

Now using (3.9) we get

$$\|y(t)\|_{\mathcal{Z}} \leqslant C(t-\tau)^{-\gamma}\|y_0\|_{\mathcal{Z}} + C\int_\tau^t (t-s)^{-\gamma}(s-\tau)^{-\gamma}\|y_0\|_{\mathcal{Y}}\, ds \leqslant C(t-\tau)^{-\gamma}\|y_0\|_{\mathcal{Z}}. \square$$

With this we have

Corollary 3.6. *Assume ξ_η^* is a global solution of (2.4) or (2.3) for $\eta = 0$. Then the linearized processes $U_\eta(t,\tau)$ and $V_\eta(t,\tau)$ in (2.11) are well defined in $\mathcal{L}(\mathcal{Z})$ and in $\mathcal{L}(\mathcal{Y})$. Even more they belong to $\mathcal{L}(\mathcal{Z},\mathcal{Y})$ and satisfy (3.9) and (3.10).*

4. Continuity results for linear and nonlinear processes under singular perturbation

In this section we provide a proof of Proposition 2.17. Throughout this subsection we keep the notation from Section 2.

Also, we will prove that the linearized processes in (2.10) and (2.11) converge as $\eta \to 0$. Note that this is also a preliminary step in order to later apply Theorem 2.20, since we will have therefore (2.16) and (2.17) satisfied.

In the next result we proceed as in Corollary 3.6 but we use that some terms are bounded independent of $t \in \mathbb{R}$.

Lemma 4.1. *Assume that Condition 2.13 and Condition 2.15 hold. Assume ξ_0^* is a global bounded solution of (2.3). Then, for each $T > 0$,*

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|V_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant C(T) \tag{4.1}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|V_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Y})} \leqslant C(T) \tag{4.2}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|V_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Z})} \leqslant C(T) \tag{4.3}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|V_\eta(t,\tau)-V_0(t,\tau)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \to 0 \text{ as } \eta\to 0. \tag{4.4}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|V_\eta(t,\tau)-V_0(t,\tau)\|_{\mathcal{L}(\mathcal{Y})} \to 0 \text{ as } \eta\to 0. \tag{4.5}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|V_\eta(t,\tau)-V_0(t,\tau)\|_{\mathcal{L}(\mathcal{Z})} \to 0 \text{ as } \eta\to 0. \tag{4.6}$$

Proof: To show (4.1) and (4.2) we use the expression of V_η given by (2.11), Condition 2.11 and Condition 2.13. Hence, if $\mathcal{W} = \mathcal{Z}$ or $\mathcal{Y}$

$$\begin{aligned}
\|V_\eta(t,\tau)y\|_\mathcal{Y} &\leqslant M(t-\tau)^{-\gamma}e^{-\beta(t-\tau)}\|y\|_\mathcal{W} + MC\int_\tau^t e^{-\beta(t-s)}(t-s)^{-\gamma}\|V_\eta(s,\tau)y\|_\mathcal{Y}ds \\
&\leqslant M(t-\tau)^{-\gamma}\|y\|_\mathcal{W} + MC\int_\tau^t (t-s)^{-\gamma}\|V_\eta(s,\tau)y\|_\mathcal{Y}ds.
\end{aligned}$$

Applying now the singular Gronwall lemma, see [27, Lemma 7.1.1], we obtain (4.1) and (4.2).

To show (4.3) we use the expression of V_η given by (2.11), so that

$$\begin{aligned}
\|V_\eta(t,\tau)y\|_\mathcal{Z} &\leqslant M(t-\tau)^{-\gamma}e^{-\beta(t-\tau)}\|y\|_\mathcal{Z} + MC\int_\tau^t (t-s)^{-\gamma}e^{-\beta(t-s)}\|V_\eta(s,\tau)y\|_\mathcal{Y}ds \\
&\leqslant M(t-\tau)^{-\gamma}\|y\|_\mathcal{Z} + MC\int_\tau^t (t-s)^{-\gamma}(s-\tau)^{-\gamma}ds\,\|y\|_\mathcal{Z} \\
&\leqslant C(T)(t-\tau)^{-\gamma}\|y\|_\mathcal{Z}
\end{aligned}$$

where we have used (4.1) inside the integral.

To show (4.4) and (4.5), we have from (2.11) that

$$V_\eta(t,\tau) - V_0(t,\tau) = e^{\mathfrak{B}_\eta(t-\tau)} - e^{\mathfrak{B}_0(t-\tau)} + \int_\tau^t (e^{\mathfrak{B}_\eta(t-s)} - e^{\mathfrak{B}_0(t-s)}) D_y f(s,\xi_0^*(s)) V_\eta(s,\tau) ds$$
$$+ \int_\tau^t e^{\mathfrak{B}_0(t-s)} D_y f(s,\xi_0^*(s))(V_\eta(s,\tau) - V_0(s,\tau)) ds,$$

and consequently, if $\mathcal{W} = \mathcal{Z}$ or $\mathcal{Y}$,

$$\begin{aligned}\|V_\eta(t,\tau) - V_0(t,\tau)\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} &\leqslant \|e^{\mathfrak{B}_\eta(t-\tau)} - e^{\mathfrak{B}_0(t-\tau)}\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} \\ &+ \int_\tau^t \|e^{\mathfrak{B}_\eta(t-s)} - e^{\mathfrak{B}_0(t-s)}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \|D_y f(s,\xi_0^*(s))\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})} \|V_\eta(s,\tau)\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} ds \\ &+ \int_\tau^t \|e^{\mathfrak{B}_0(t-s)}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \|D_y f(s,\xi_0^*(s))\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})} \|V_\eta(s,\tau) - V_0(s,\tau)\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} ds.\end{aligned}$$

Taking into account the uniform estimates of the linear semigroups given by Condition 2.11, Condition 2.13 and (4.1), we get for $0 \le t-\tau \leqslant T$,

$$\begin{aligned}\|V_\eta(t,\tau) - V_0(t,\tau)\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} \leqslant & \rho(\eta,T)(t-\tau)^{-\gamma} + C\rho(\eta,T) \int_\tau^t (t-s)^{-\gamma}(s-\tau)^{-\gamma} ds \\ & + \int_\tau^t M(t-s)^{-\gamma} C \|V_\eta(s,\tau) - V_0(s,\tau)\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} ds \\ \leqslant & (1 + CT^{1-\gamma})\rho(\eta,T)(t-\tau)^{-\gamma} \\ & + MC \int_\tau^t (t-s)^{-\gamma} \|V_\eta(s,\tau) - V_0(s,\tau)\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} ds.\end{aligned}$$

The result follows applying the singular Gronwall's lemma to the last inequality.

To show (4.6) we proceed in a similar manner. □

In a completely similar way we have that

Lemma 4.2. *Assume we have a bounded and global solution $\xi_0^*(\cdot)$ of (2.3) and a sequence of bounded global solutions $\xi_\eta^*(\cdot)$ of (2.4) such that*

$$\sup_{t\in\mathbb{R}} \|\xi_\eta^*(t) - \xi_0^*(t)\|_{\mathcal{Y}} \to 0 \text{ as } \eta \to 0.$$

Then, if U_η is the linear process generated by (2.8), which is defined in (2.10), we have

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|U_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant C(T) \tag{4.7}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|U_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Y})} \leqslant C(T) \tag{4.8}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|U_\eta(t,\tau)\|_{\mathcal{L}(\mathcal{Z})} \leqslant C(T) \tag{4.9}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|U_\eta(t,\tau) - U_0(t,\tau)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \to 0 \text{ as } \eta \to 0. \tag{4.10}$$

$$\sup_{\tau\in\mathbb{R}} \sup_{0<t-\tau\leqslant T} (t-\tau)^\gamma \|U_\eta(t,\tau) - U_0(t,\tau)\|_{\mathcal{L}(\mathcal{Y})} \to 0 \text{ as } \eta \to 0. \tag{4.11}$$

$$\sup_{\tau\in\mathbb{R}}\sup_{0<t-\tau\leqslant T}(t-\tau)^{\gamma}\|U_\eta(t,\tau)-U_0(t,\tau)\|_{\mathcal{L}(\mathcal{Z})}\to 0 \ \ as\ \eta\to 0. \tag{4.12}$$

Recall that $V_0 = U_0$. Then we are now ready for the

Proof of Proposition 2.17. Taking into account (2.7), we have, for $y\in\mathcal{Y}$ with $\|y\|_{\mathcal{Y}}\leqslant r$,

$$\begin{aligned}\|T_\eta(t,\tau)y-T_0(t,\tau)y\|_{\mathcal{Y}} &\leq \|(e^{\mathfrak{B}_\eta(t-\tau)}-e^{\mathfrak{B}_0(t-\tau)})y\|_{\mathcal{Y}}\\ &+\int_\tau^t \|e^{\mathfrak{B}_\eta(t-s)}-e^{\mathfrak{B}_0(t-s)}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})}\|f(s,T_\eta(s,\tau)y)\|_{\mathcal{Z}}\,ds,\\ &+\int_\tau^t \|e^{\mathfrak{B}_0(t-s)}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})}\|f(s,T_\eta(s,\tau)y)-f(s,T_0(s,\tau)y)\|_{\mathcal{Z}}\,ds.\end{aligned} \tag{4.13}$$

Using Condition 2.11, that f is bounded and globally Lipschitz, Condition 2.13 and Condition 2.15 we have that

$$\begin{aligned}\|T_\eta(t,\tau)y-T_0(t,\tau)y\|_{\mathcal{Y}} \leqslant \rho(\eta,T)(t-\tau)^{-\gamma}r&+\int_\tau^t\rho(\eta,T)(t-s)^{-\gamma}K\,ds,\\ &+\int_\tau^t M(t-s)^{-\gamma}L\|T_\eta(s,\tau)y-T_0(s,\tau)y\|_{\mathcal{Y}}\,ds,\end{aligned}$$

where K is a bound of the nonlinearity f and L is the Lipschitz constant of f with respect to the second variable. Now we easily get

$$\begin{aligned}\|T_\eta(t,\tau)y-T_0(t,\tau)y\|_{\mathcal{Y}} &\leqslant C(T,r)\rho(\eta,T)(t-\tau)^{-\gamma}\\ &+ML\int_\tau^t(t-s)^{-\gamma}\|T_\eta(s,\tau)y-T_0(s,\tau)y\|_{\mathcal{Y}}\,ds,\end{aligned}$$

for some constant $C(T,r)$. Hence, applying the singular Gronwall's lemma we obtain (2.17). □

5. **Roughness of singular exponential dichotomy**

In this section we provide a proof of the general result Theorem 2.20 and introduce some results on dichotomies.

For the problems considered in Section 2, see (2.3), (2.4), these tools will allow us to transfer information from the auxiliary space $\mathcal{Z}$ to the phase space $\mathcal{Y}$. We note that some of the results presented here are generalizations of results that appear in [27] for the case when $\mathcal{Y}$ is a fractional power space in $\mathcal{Z}$ associated to the linear part of the equations and the unbounded operator is fixed. Our aim is to provide a proof of Theorem 2.20 which extends the results on roughness of dichotomy of [27] for linear evolution processes satisfying the singular perturbation property (2.17). To this end we start with the definition of discrete dichotomy

Definition 5.1. *We say that a family $\{T_n\}_{n\in\mathbb{Z}}\subset\mathcal{L}(\mathcal{Z})$ has discrete dichotomy (with constant M and exponent ω) if there are positive constants M,ω and projections $\{Q_n\}_{n\in\mathbb{Z}}\subset\mathcal{L}(\mathcal{Z})$ such that*

i) $T_nQ_n = Q_{n+1}T_n$, $n\in\mathbb{Z}$,

ii) *for* $n \geqslant m$, $\|T_{n,m}(I-Q_m)\|_{\mathcal{L}(\mathcal{Z})} \leqslant Me^{-\omega(n-m)}$, *where* $T_{n,m} = T_{n-1} \circ T_{n-2} \cdots \circ T_m$ *for* $n > m$ *and* $T_{n,n} = I$.

iii) *for* $m > n$, $T_{m,n} : R(Q_n) \to R(Q_m)$ *is an isomorphism with inverse* $T_{n,m} : R(Q_m) \to R(Q_n)$, *and* $\|T_{n,m}Q_m\|_{\mathcal{L}(\mathcal{Z})} \leqslant Me^{\omega(n-m)}$.

If $\{T_n\}_{n\in\mathbb{Z}}$ has exponential dichotomy with projections $\{Q_n\}_{n\in\mathbb{Z}}$, constant M and exponent ω, define

$$G_{n,m} = \begin{cases} T_{n,m}(I-Q_m), & n \geqslant m \\ -T_{n,m}Q_m, & n < m. \end{cases} \tag{5.1}$$

It is easy to see that

- $\|Q_n\|_{\mathcal{L}(\mathcal{Z})} \leqslant M$ for all $n \in \mathbb{Z}$ and
- $\|G_{n,m}\|_{\mathcal{L}(\mathcal{Z})} \leqslant Me^{-\omega|n-m|}$ for all $n \geqslant m \in \mathbb{Z}$.

As an consequence of Definition 2.10 and Definition 5.1 we have the following result (see [27, Page 229])

Theorem 5.2. *Suppose that* $\{W(t,s) : t \geqslant s \in \mathbb{R}\}$ *is an evolution process in* $\mathcal{Z}$ *which has exponential dichotomy with projections* $\{Q(t) : t \in \mathbb{R}\}$, *constant* M *and exponent* ω. *For each* $\ell > 0$ *and* $t_0 \in \mathbb{R}$,

$$\{T_n\}_{n\in\mathbb{Z}} := \{W(t_0+(n+1)\ell, t_0+n\ell)\}_{n\in\mathbb{Z}}$$

has discrete dichotomy with projections $\{Q_n = Q(t_0+n\ell) : n \in \mathbb{Z}\}$, *constant* M *and exponent* $\omega\ell$.

We also have

Theorem 5.3. *If* $\{T_n\}_{n\in\mathbb{Z}}$ *has discrete dichotomy and* $\{y_n\}_{n\in\mathbb{Z}}$ *is a bounded sequence in* $\mathcal{Z}$, *then there is a unique bounded solution* $\{x_n\}_{n\in\mathbb{Z}}$ *of*

$$x_{n+1} = T_n x_n + y_n, \quad n \in \mathbb{Z}, \tag{5.2}$$

and this solution is given by

$$x_n = \sum_{k=-\infty}^{\infty} G_{n,k+1}\, y_k, \tag{5.3}$$

where $G_{n,m}$ *is given by* (5.1).

Proof: To see that a bounded solution of (5.2) is given by (5.3) simply observe that

$$x_n = T_{n,m}x_m + \sum_{k=m}^{n-1} T_{n,k+1}y_k$$

from this $(I-Q_n)x_n = T_{n,m}(I-Q_m)x_m + \sum_{k=m}^{n-1} T_{n,k+1}(I-Q_{k+1})y_k,\ n \geqslant m$. Taking the limit as $m \to -\infty$ we have that

$$(I-Q_n)x_n = \sum_{k=-\infty}^{n-1} T_{n,k+1}(I-Q_{k+1})y_k.$$

On the other hand $Q_r x_r = T_{r,n} Q_n x_n + \sum_{k=n}^{r-1} T_{r,k+1} Q_{k+1} y_k,\ r \geqslant n$. Hence

$$T_{n,r} Q_r x_r = Q_n x_n + \sum_{k=n}^{r-1} T_{n,k+1} Q_{k+1} y_k,\ r > n.$$

Now, taking the limit as $r \to \infty$, we arrive at

$$Q_n x_n = -\sum_{k=n}^{\infty} T_{n,k+1} Q_{k+1} y_k.$$

The existence of a bounded solution is done simply inspecting that $\{x_n\}_{n\in\mathbb{Z}}$ given by (5.3) satisfies (5.2).□

The next results states the continuity of the discrete projections.

Theorem 5.4. *Suppose that $\{T_n^{(k)}\}_{n\in\mathbb{Z}}$ has discrete dichotomy in $\mathcal{Z}$ with projections $\{Q_n^{(k)}\}_{n\in\mathbb{Z}}$, constant M and exponent ω for $k=1,2$. If $\|T_n^{(1)} - T_n^{(2)}\|_{\mathcal{L}(\mathcal{Z})} \leqslant \epsilon$ for all $n \in \mathbb{Z}$, then*

$$\|Q_n^{(1)} - Q_n^{(2)}\|_{\mathcal{L}(\mathcal{Z})} \leqslant \frac{2M^2}{1-e^{-\omega}}\,\epsilon.$$

Furthermore, let $\mathcal{Y} \subset \mathcal{Z}$ be Banach spaces and let $\mathcal{W}$ be either $\mathcal{Z}$ or $\mathcal{Y}$. If in (5.1) we have

$$\|G_{n,m}^{(1)}\|_{\mathcal{L}(\mathcal{Y})} \leqslant M e^{-\omega|n-m|} \text{ for all } n \geqslant m \in \mathbb{Z}$$

and $\|T_n^{(1)} - T_n^{(2)}\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} \leqslant \epsilon$ for all $n \in \mathbb{Z}$ then,

$$\|Q_n^{(1)} - Q_n^{(2)}\|_{\mathcal{L}(\mathcal{W},\mathcal{Y})} \leqslant \frac{2M^2}{1-e^{-\omega}}\,\epsilon.$$

Proof: Define $x_n^{(k)} = G_{n,m}^{(k)} z$ for $z \in \mathcal{Z}$. Note that

$$x_{n+1}^{(1)} - T_n^{(2)} x_n^{(1)} = \begin{cases} T_n^{(1)} x_n^{(1)} - T_n^{(2)} x_n^{(1)}, & n \neq m-1 \\ T_n^{(1)} x_n^{(1)} - T_n^{(2)} x_n^{(1)} + z, & n = m-1 \end{cases}$$

and

$$x_{n+1}^{(2)} - T_n^{(2)} x_n^{(2)} = \begin{cases} 0, & n \neq m-1 \\ z, & n = m-1. \end{cases}$$

Since $z_n = x_n^{(1)} - x_n^{(2)}$ satisfy $z_{n+1} = T_n^{(2)} z_n + y_n$ where $y_n = (T_n^{(1)} - T_n^{(2)}) x_n^{(1)}$, from the definition of $x_n^{(1)}$ and from the hypothesis on $T_n^{(1)} - T_n^{(2)}$ we have $\{y_n\}_{n\in\mathbb{Z}}$ is bounded and from Theorem 5.3 we have that

$$z_n = \sum_{k=-\infty}^{\infty} G_{n,k+1}^{(2)} (T_n^{(1)} - T_n^{(2)}) G_{k,m}^{(1)} z$$

and consequently

$$\|z_n\|_{\mathcal{Z}} \leqslant \sum_{k=-\infty}^{\infty} M^2 e^{-\omega|n-k-1|} e^{-\omega|k-m|} \|T_n^{(1)} - T_n^{(2)}\|_{\mathcal{L}(\mathcal{Z})} \|z\|_{\mathcal{Z}} \leqslant \frac{2M^2}{1-e^{-\omega}}\,\epsilon\,\|z\|_{\mathcal{Z}}.$$

To conclude the proof of the first statement, simply note that

$$z_m = x_m^{(1)} - x_m^{(2)} = (G_{m,m}^{(1)} - G_{m,m}^{(2)})z = (Q_m^{(2)} - Q_m^{(1)})z.$$

The remaining statement has an identical proof.□

The next is reciprocal to Theorem 5.2 establishing that discrete dichotomy implies exponential dichotomy (for the case when the process is not singular at the initial time this result can be found in [27, Exercise 10, page 229]). The singular case, which is needed for many applications, is proved next

Theorem 5.5. *Assume that the evolution process* $\{W(t,\tau) : t \geqslant \tau \in \mathbb{R}\} \subset \mathcal{L}(\mathcal{Z})$ *satisfies*

$$L_\ell = \sup_{0\leqslant t-\tau\leqslant \ell} (t-\tau)^\gamma \|W(t,\tau)\|_{\mathcal{L}(\mathcal{Z})} < \infty \tag{5.4}$$

for some $\ell > 0$. *Assume that there are* $M, \omega > 0$ *such that, for each* $t_0 \in \mathbb{R}$, $\{W(t_0 + (n+1)\ell, t_0 + n\ell)\}_{n\in\mathbb{Z}}$ *has discrete dichotomy with constant* M *and exponent* ω. *Hence, the evolution process*

$$\{W(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$$

has exponential dichotomy with constant KM *and exponent* ω, *where the constant* K *is given by* $K = \sup_{\ell\leqslant t-\tau\leqslant 2\ell}\{e^{\omega(t-\tau)}\|W(t,\tau)\|_{\mathcal{L}(\mathcal{Z})}, (t-\tau-\ell)^\gamma \|W(t,\tau+\ell)\|_{\mathcal{L}(\mathcal{Z})} e^{\omega(t-\tau)}\}$.

Proof: Let $\{Q_n(t_0)\}_{n\in\mathbb{Z}}$ be the family of projections associated to the discrete dichotomy of $\{W(t_0 + (n+1)\ell, t_0 + n\ell)\}_{n\in\mathbb{Z}}$ and define $Q(t_0) = Q_0(t_0)$.

Let $\{T_n(t)\}_{n\in\mathbb{Z}} := \{W(t + (n+1)\ell, t + n\ell)\}_{n\in\mathbb{Z}}$, for each $t \in \mathbb{R}$. From the definition of the family of projections $\{Q_n(t + k\ell)\}_{n\in\mathbb{Z}}$, the family of linear operators

$$\{T_n(t + k\ell)\}_{n\in\mathbb{Z}} = \{T_{n+k}(t)\}_{n\in\mathbb{Z}}$$

has discrete dichotomy with projections $\{Q_n(t+k\ell)\}_{n\in\mathbb{Z}}$ and with projections $\{Q_{n+k}(t)\}_{n\in\mathbb{Z}}$. From the uniqueness of projections $Q_{n+k}(t) = Q_n(t + k\ell)$ for all $n \in \mathbb{Z}$ and in particular $Q_k(t) = Q_0(t + k\ell)$.

Next we show that, for $n > k$,

$$Q(t + n\ell)W(t + n\ell, t + k\ell) = W(t + n\ell, t + k\ell)Q(t + k\ell). \tag{5.5}$$

In fact, first note that

$$\begin{aligned} W(t + k\ell &+ (n-k)\ell, t + k\ell) \\ &= W(t + k\ell + (n-k)\ell, t + k\ell + (n-k-1)\ell) \cdots W(t + k\ell + \ell, t + k\ell). \end{aligned}$$

Then,

$$\begin{aligned} W(t + n\ell, t + k\ell)Q(t + k\ell) &= W(t + k\ell + (n-k)\ell, t + k\ell)Q_0(t + k\ell) \\ &= Q_{n-k}(t + k\ell)(W(t + k\ell + (n-k)\ell, t + k\ell) \\ &= Q(t + n\ell)W(t + n\ell, t + k\ell) \end{aligned}$$

and (5.5) is proved. Now, let us show that

$$\|W(t,s)Q(s)\|_{\mathcal{L}(\mathcal{Z})} \leqslant KMe^{-\omega(t-s)}, \quad t \geqslant s. \tag{5.6}$$

For $t \geqslant s$ choose $n \in \mathbb{N}$ such that $s + n\ell \leqslant t < s + (n+1)\ell$. Then, if $n \geqslant 1$,

$$\begin{aligned}\|W(t,s)(I-Q(s))\|_{\mathcal{L}(\mathcal{Z})} &\leqslant \|W(t,s+(n-1)\ell)\|_{\mathcal{L}(\mathcal{Z})}\|W(s+(n-1)\ell,s)(I-Q(s))\|_{\mathcal{L}(\mathcal{Z})}\\ &\leqslant Me^{-\omega(n-1)\ell}\|W(t,s+(n-1)\ell)\|_{\mathcal{L}(\mathcal{Z})}\\ &= Me^{\omega(t-s-(n-1)\ell)}\|W(t,s+(n-1)\ell)\|_{\mathcal{L}(\mathcal{Z})}e^{-\omega(t-s)}\\ &\leqslant KMe^{-\omega(t-s)},\end{aligned}$$

if $n = 0$,

$$\|W(t,s)(I-Q(s))\|_{\mathcal{L}(\mathcal{Z})} \leqslant KM(t-s)^{-\gamma}e^{-\omega(t-s)}.$$

Joining these two cases we have that (5.6) is proved.

Assume that $z \in R(Q(s))$ and $t \leqslant s$. Define

$$W(t,s)z = W(t,s+n\ell)(W(s,s+n\ell)\big|_{R(Q(s+n\ell))})^{-1}z$$

where $n \in \mathbb{Z}$ is chosen such that $s+(n+1)\ell > t \geqslant s+n\ell$. We show that, with this definition,

$$\|W(t,s)Q(s)\|_{\mathcal{L}(\mathcal{Z})} \leqslant KMe^{\omega(t-s)}, \quad t \leqslant s.$$

In fact,

$$\begin{aligned}\|W(t,s)Q(s)\|_{\mathcal{L}(\mathcal{Z})} &\leqslant \|W(t,s+(n-1)\ell)\|_{\mathcal{L}(\mathcal{Z})}\|(W(s,s+(n-1)\ell)\big|_{R(Q(s))})^{-1}Q(s+(n-1)\ell)\|_{\mathcal{L}(\mathcal{Z})}\\ &\leqslant Me^{\omega(n-1)\ell}\|W(t,s+(n-1)\ell)\|_{\mathcal{L}(\mathcal{Z})}\\ &= Me^{-\omega(t-s-(n-1)\ell)}\|W(t,s+n\ell)\|_{\mathcal{L}(\mathcal{Z})}e^{\omega(t-s)}\\ &\leqslant KMe^{\omega(t-s)}.\end{aligned}$$

Let us show that

$$N(Q(t_0)) = \{z \in \mathcal{Z} : [t_0,\infty) \ni t \mapsto W(t,t_0)z \in \mathcal{Z} \text{ is bounded}\}. \tag{5.7}$$

If $z \in N(Q(t_0))$, then $Q(t_0)z = 0$ and

$$\|W(t,t_0)z\|_{\mathcal{Z}} = \|W(t,t_0)(I-P(t_0))z\|_{\mathcal{Z}} \leqslant KMe^{-\omega(t-t_0)}$$

therefore, $[t_0,\infty) \ni t \mapsto W(t,t_0)z \in \mathcal{Z}$ is bounded. On the other hand, if $z \notin N(Q(t_0))$, then

$$W(t,t_0)z = W(t,t_0)(I-Q(t_0))z + W(t,t_0)Q(t_0)z.$$

Note that, $W(t,t_0)(I-Q(t_0))z$ is bounded for $t \geqslant t_0$. Also,

$$\begin{aligned}\|Q(t_0)\|_{\mathcal{L}(\mathcal{Z})} &= \|W(t_0,t_0+n\ell)W(t_0+n\ell,t_0)Q(t_0)\|_{\mathcal{L}(\mathcal{Z})}\\ &\leqslant Me^{-\omega n\ell}\|W(t_0+n\ell,t_0)Q(t_0)\|_{\mathcal{L}(\mathcal{Z})}\end{aligned}$$

or equivalently $\|W(t_0+n\ell,t_0)Q(t_0)z\|_{\mathcal{L}(\mathcal{Z})} \geqslant M^{-1}e^{\omega n\ell}\|Q(t_0)z\|_{\mathcal{L}(\mathcal{Z})}$. This proves that the function $[t_0,\infty) \ni t \to W(t,t_0)Q(t_0)z$ is unbounded. Hence, whenever $z \notin N(Q(t_0))$ we have that $[t_0,\infty) \ni t \mapsto W(t,t_0)z$ is unbounded. This completes the proof of (5.7).

Next we show that

$$W(t,t_0)N(Q(t_0)) \subset N(Q(t)), \quad t \geqslant t_0.$$

In fact, from (5.7) we have that, if $z \in N(Q(t_0))$, then $[t_0,\infty) \ni s \mapsto W(s,t_0)s \in \mathcal{Z}$ is bounded and consequently $[t,\infty) \ni s \mapsto W(s,t)W(t,t_0)z \in \mathcal{Z}$ is bounded and $W(t,t_0)z \in N(Q(t))$.

To see that $W(t,t_0)\big|_{R(Q(t_0))} : R(Q(t_0)) \to \mathcal{Z}$ is injective, $t \geqslant t_0$ assume $z \in R(Q(t_0))$ is such that $W(t,t_0)z = 0$. Choosing $n \in \mathbb{N}$ such that $t_0 + n\ell \geqslant t$, we have that $W(t_0 + n\ell, t_0)z = 0$ and consequently $z = 0$. Proceeding exactly in the same manner we obtain that $W(s,t)\big|_{W(t,t_0)R(Q(t_0))} : W(t,t_0)R(Q(t_0)) \to \mathcal{Z}$ is injective.

Now, if $z \in R(Q(t_0))$ define $y : (-\infty, t_0] \to \mathcal{Z}$ by $y(t) = W(t, t_0+n\ell)W(t_0+n\ell, t_0)z$ where $n \in \mathbb{Z}$ is chosen such that $t_0+n\ell \leqslant t \leqslant t_0+(n+1)\ell$. Clearly $y : (-\infty, t_0] \to \mathcal{Z}$ is continuous and $W(s,t)y(t) = y(s)$ for all $t_0 \geqslant s \geqslant t$. Furthermore, $y(t_0 + n\ell) = W(t_0 + n\ell, t_0)x \in R(Q(t_0 + n\ell))$ for all negative integer n.

Now we prove that

$$R(Q(t_0)) = \{z \in \mathcal{Z} : \text{there is a bounded continuous function } y : (-\infty, t_0] \to \mathcal{Z} \\ \text{such that } y(t_0) = z \text{ and } W(s,t)y(t) = y(s) \text{ for all } t_0 \geqslant s \geqslant t\} \tag{5.8}$$

It is clear that, if $z \in R(Q(t_0))$, there is a bounded continuous function $y : (-\infty, t_0] \to \mathcal{Z}$ such that $y(t_0) = x$ and $W(s,t)y(t) = y(s)$ for all $t_0 \geqslant s \geqslant t$.

Assume that $z \notin R(Q(t_0))$ and that there is a function $y : (-\infty, t_0] \to \mathcal{Z}$ such that $y(t_0) = z$ and $W(s,t)y(t) = y(s)$ for all $t_0 \geqslant s \geqslant t$. Clearly $W(t_0, t_0 + n\ell)y(t_0 + n\ell) = z$ and

$$\begin{aligned} Q(t_0)z + (I - Q(t_0))z = {} & W(t_0, t_0 + n\ell)Q(t_0 + n\ell)y(t_0 + n\ell) \\ & + W(t_0, t_0 + n\ell)(I - Q(t_0 + n\ell))y(t_0 + n\ell) \end{aligned}$$

Consequently

$$W(t_0, t_0 + n\ell)Q(t_0 + n\ell)y(t_0 + n\ell) = Q(t_0)z$$

and

$$W(t_0, t_0 + n\ell)(I - Q(t_0 + n\ell))y(t_0 + n\ell) = (I - Q(t_0))z.$$

Hence $Q(t_0 + n\ell)y(t_0 + n\ell) = W(t_0 + n\ell, t_0)Q(t_0)z$ remains bounded as $n \to -\infty$ whereas

$$\|(I - Q(t_0))z\|_{\mathcal{Z}} \leqslant Me^{\omega n\ell}\|(I - Q(t_0 + n\ell))y(t_0 + n\ell)\|_{\mathcal{Z}}$$

and $\|(I - Q(t_0 + n\ell))y(t_0 + n\ell)\|_{\mathcal{Z}} \geqslant M^{-1}e^{-\omega n\ell}\|(I - Q(t_0))z\|_{\mathcal{Z}} \overset{n\to-\infty}{\longrightarrow} \infty$. Showing that $y : (-\infty, t_0] \to \mathcal{Z}$ is not bounded and completing the proof of (5.8).

It remains to prove that $W(t,t_0)(R(Q(t_0))) = R(Q(t))$ for all $t \geqslant 0$. In fact, if $z \in R(Q(t_0))$ let $y : (-\infty, t_0] \to \mathcal{Z}$ such that $y(t_0) = z$ and $W(s,r)y(r) = y(s)$ for all $t_0 \geqslant s \geqslant r$. Defining $\bar{y} : (-\infty, t] \to \mathcal{Z}$ by $\bar{y}(r) = y(r)$, $r \leqslant t_0$ and $\bar{y}(r) = W(r, t_0)z$ for $r \in [t_0, t]$ we have that $\bar{y}(t) = W(t,t_0)z$ and $W(s,r)\bar{y}(r) = \bar{y}(s)$ for all $t \geqslant s \geqslant r$. Hence $W(t,t_0) \in R(Q(t))$. On the other hand, if $z \in R(Q(t))$, for $t \geqslant t_0 \geqslant t + n\ell$ we have that $z = W(t_0, t+n\ell)W(t+n\ell, t)z \in R(Q(t_0))$ and $W(t,t_0)z = z$.

It follows that

$$W(t,t_0)\big|_{R(Q(t_0))} : R(Q(t_0)) \to R(Q(t))$$

is an isomorphism. This completes the proof. □

Before we can prove Theorem 2.20 we need to recall the following result (see [27, Theorem 7.6.7])

Theorem 5.6. *Suppose that $\{T_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(\mathcal{Z})$ has discrete dichotomy with constant M and exponent ω. If $M_1 > M$ and $\omega_1 < \omega$, there exists $\epsilon > 0$ (depending only on M, M_1, θ and θ_1) such that any family $\{S_n\}_{n\in\mathbb{Z}}$ such that $\sup_{n\in\mathbb{Z}} \|S_n - T_n\|_{\mathcal{L}(\mathcal{Z})} < \epsilon$ has discrete dichotomy with constant M_1 and exponent ω_1.*

Now we are ready to prove Theorem 2.20.

Proof of Theorem 2.20. The idea of the proof is the following: First we pass from the continuous processes $\{W_\eta(t,s) : t \geqslant s\}$ to discrete process $\{W_\eta(t_{n+1}, t_n) : n \in \mathbb{Z}\}$ and using Theorem 5.2 we obtain discrete dichotomy for $\{W_0(t_{n+1}, t_n) : n \in \mathcal{Z}\}$. Next, we use Theorem 5.6 to obtain discrete dichotomy for $\{W_\eta(t_{n+1}, t_n) : n \in \mathcal{Z}\}$ with η suitably small. Finally, we apply Theorem 5.6 to obtain exponential dichotomy for $\{W_\eta(t_{n+1}, t_n) : n \in \mathcal{Z}\}$ with η suitably small. The continuity of projections will then follow from Theorem 5.4.

Let $T = 2\ell$ and $\ell > 0$ be such that $Me^{-(\omega_0-\omega)\ell} < 1$. For each $t_0 \in \mathbb{R}$ let $t_n = t_0 + n\ell$, $n \in \mathbb{N}$. It is clear that

$$\{W_0(t_{n+1}, t_n)\}_{n\in\mathbb{Z}}$$

has discrete dichotomy with constants M and exponent $\omega > 0$.

Note that, given $\epsilon >$ there exists $\eta_\epsilon > 0$ such that

$$\sup_{\eta\in[0,\eta_\epsilon]} \|W_\eta(t_{n+1}, t_n) - W_0(t_{n+1}, t_n)\|_{\mathcal{L}(\mathcal{Z})} \leqslant \epsilon$$

for all $t_0 \in \mathbb{R}$. It follows from Theorem 5.6 that, given $M_1 > M$ and $\omega < \omega_0$, there exists $\epsilon > 0$ such that $\{W_\eta(t_{n+1}, t_n)\}_{n\in\mathbb{Z}}$ has discrete dichotomy with constant M_1 and exponent ω_1 for all $\eta \in [0, \eta_\epsilon]$.

It follows from Theorem 5.5 that $\{W_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has exponential dichotomy with constant $M := KM_1$, ω, for all $\eta \leqslant \eta_\epsilon$.

To prove the continuity of projections let $0 < \omega < \omega_0$ and $M_1 > M$ and choose $\eta_0 > 0$ such that $\{W_\eta(t,s) : t \geqslant s \in \mathbb{R}\}$ has exponential dichotomy with constant $M := KM_1$ and exponent ω for all $0 \leqslant \eta \leqslant \eta_0$. Furthermore, let $0 < \eta_\epsilon \leqslant \eta_0$ be such that $\|W_\eta(t + \ell, t) - W_0(t + \ell, t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant \epsilon$. For each $t_0 \in \mathbb{R}$, if $T_n^{(2)} = W_\eta(t_0 + (n+1)\ell, t_0 + n\ell)$ and $T_n^{(1)} = W_0(t_0 + (n+1)\ell, t_0 + n\ell)$, we have from Theorem 5.2 that the conditions of Theorem 5.4 are satisfied and

$$\|Q_\eta(t_0 + n\ell) - Q_0(t_0 + n\ell)\|_{\mathcal{L}(\mathcal{Z})} \leqslant \frac{2M^2}{1 - e^{-\omega}}\,\epsilon, \quad \forall\, n \in \mathbb{Z}.$$

Since that holds independently of t_0 we have the result.

□

The following result allow us to transfer information from the auxiliary space $\mathcal{Z}$ to the phase space $\mathcal{Y}$.

Theorem 5.7. *Assume that $\{W(t,s) : t \geqslant s\} \subset \mathcal{L}(\mathcal{Z})$ has exponential dichotomy with exponent ω and constant M and a family of projections $\{Q(t) : t \in \mathbb{R}\} \subset \mathcal{L}(\mathcal{Z})$. Assume also that, $\mathcal{Y} \subset \mathcal{Z}$ and for some $\bar{M} \geqslant 1$,*

$$\|W(t,\tau)z\|_{\mathcal{Y}} \leqslant \bar{M}(t-\tau)^{-\gamma}\|z\|_{\mathcal{Z}}, \quad 0 < t - \tau \leqslant 1.$$

Then, for each $\omega_1 < \omega$, there is a constant $M_1 > M$ such that, for each $z \in \mathcal{Z}$,

$$\begin{aligned} &\|W(t,s)(I-Q(s))z\|_{\mathcal{Y}} \leqslant M_1 \max\{1,(t-s)^{-\gamma}\}e^{-\omega_1(t-s)}\|z\|_{\mathcal{Z}}, \quad t>s \\ &\|W(t,s)Q(s)z\|_{\mathcal{Y}} \leqslant M_1 e^{\omega_1(t-s)}\|z\|_{\mathcal{Z}}, \quad t \leqslant s. \end{aligned} \tag{5.9}$$

In particular, $Q(t)\mathcal{Z} \subset \mathcal{Y}$ with $\|Q(t)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant M_1$, for all $t \in \mathbb{R}$ and $Q(t)$ and $I-Q(t)$ are continuous projections in $\mathcal{L}(\mathcal{Y})$ for every $t \in \mathbb{R}$.

Proof: Using property *ii)* of Definition 2.10, we have that, for $t \leqslant s$,

$$\begin{aligned} \|W(t,s)Q(s)z\|_{\mathcal{Y}} &= \|W(t,t-1)W(t-1,s)Q(s)z\|_{\mathcal{Y}} \\ &\leqslant \bar{M}(t-(t-1))^{-\gamma}\|W(t-1,s)Q(s)z\|_{\mathcal{Z}} \leqslant \bar{M}Me^{\omega(t-s)}\|z\|_{\mathcal{Z}} \\ &\leqslant M_1 e^{\omega_1(t-s)}\|z\|_{\mathcal{Z}}. \end{aligned}$$

On the other hand, for $t \geqslant s+2$, we have that

$$\begin{aligned} \|W(t,s)(I-Q(s))z\|_{\mathcal{Y}} &= \|W(t,t-1)W(t-1,s)(I-Q(s))z\|_{\mathcal{Y}} \\ &\leqslant \bar{M}(t-(t-1))^{-\gamma}\|W(t-1,s)(I-Q(s))z\|_{\mathcal{Z}} \\ &\leqslant \bar{M}Me^{-\omega(t-s)}\|z\|_{\mathcal{Z}} \leqslant M_1 e^{-\omega_1(t-s)}\|z\|_{\mathcal{Z}}, \end{aligned}$$

where we have used that $\omega_1 < \omega$. Whereas for $s < t \leqslant s+2$ we have that

$$\begin{aligned} \|W(t,s)(I-Q(s))z\|_{\mathcal{Y}} &\leqslant \bar{M}(t-s)^{-\gamma}\|(I-Q(s))z\|_{\mathcal{Z}} \leqslant \bar{M}M(t-s)^{-\gamma}\|z\|_{\mathcal{Z}} \\ &\leqslant M_1(t-s)^{-\gamma}e^{-\omega_1(t-s)}\|z\|_{\mathcal{Z}}, \end{aligned}$$

From this (5.9) follows.

In particular, taking $t=s$ in the second estimate in (5.9) we get that $Q(t)\mathcal{Z} \subset \mathcal{Y}$ with $\|Q(t)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant M_1$. This in turn implies $\|Q(t)\|_{\mathcal{L}(\mathcal{Y})} \leqslant CM_1$ and the result follows.□

Now, lemma 4.2, and Theorem 5.7 allow to get Theorem 2.20 in space $\mathcal{Y}$. In particular,

$$\sup_{t\in\mathbb{R}} \|Q_\eta(t) - Q_0(t)\|_{\mathcal{L}(\mathcal{Y})} \xrightarrow{\eta\to 0} 0. \tag{5.10}$$

6. **Existence and continuity of global hyperbolic solutions**

In this section we prove that near a hyperbolic and bounded global solution of (2.3) there exists a unique global bounded solution of (2.4) which is also hyperbolic. In particular, we prove Theorem 2.18.

Suppose that $\xi_0^*(\cdot): \mathbb{R} \to \mathcal{Y}$ is a hyperbolic and bounded global solution for (2.3). Observe that equation (2.3) can be rewritten as

$$\begin{cases} \dot{y} = \mathcal{A}_0(t)y + h_0(t,y) \\ y(\tau) = y_0 \end{cases} \tag{6.1}$$

where $\mathcal{A}_0(t) = \mathfrak{B}_0 + D_y f(t,\xi_0^*(t))$ and $h_0(t,z) = f(t,z) - D_y f(t,\xi_0^*(t))z$. Then $\mathcal{A}_0(t)$ generates the linear evolution process $\{U_0(t,\tau): t \geqslant \tau \in \mathbb{R}\}$ as in (2.10) which, by assumption, has an exponential dichotomy in $\mathcal{Z}$, with constant M_0, exponent ω_0 and projections $\{Q_0(t)\}_{t\in\mathbb{R}}$.

The next result shows that for suitably small η, (2.4) has a unique global hyperbolic solution ξ_η^* near ξ_0^*. In particular, with $\eta = 0$, ξ_0^* is an isolated hyperbolic bounded solution of (2.3).

Note that a key argument below is to describe the solutions of problem (2.4) using the auxiliary process $V_\eta(t,\tau)$ in (2.11).

Theorem 6.1. *Let $\xi_0^*(\cdot)$ be a hyperbolic and bounded global solution of (2.3) and assume that Condition 2.11, Condition 2.13 and Condition 2.15 hold. Then, there exist $\epsilon, \eta_0 > 0$ such that, for each $0 \leqslant \eta < \eta_0$ there exists a unique bounded global solution $\xi_\eta^* : \mathbb{R} \to \mathcal{Y}$ of (2.4) with $\sup_{t\in\mathbb{R}} \|\xi_\eta^*(t) - \xi_0^*(t)\|_{\mathcal{Y}} \leqslant \epsilon$. Moreover, this solution satisfies*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|\xi_\eta^*(t) - \xi_0^*(t)\|_{\mathcal{Y}} = 0. \tag{6.2}$$

Furthermore, $\xi_\eta^* : \mathbb{R} \to \mathcal{Y}$ is hyperbolic for all suitably small η.

Moreover if U_η, $0 \leqslant \eta \leqslant 1$, denotes the linear process associated to (2.8), which is given by the integral expression (2.10), they have exponential dichotomies in $\mathcal{Z}$ with exponents and constants independent of η and if Q_η are the associated projections, see Definition 2.10, for $\mathcal{W} = \mathcal{Z}$ or $\mathcal{Y}$, we have

$$\sup_{t\in\mathbb{R}} \|Q_\eta(t) - Q_0(t)\|_{\mathcal{L}(\mathcal{W})} \to 0, \text{ as } \eta \to 0. \tag{6.3}$$

Proof: First note that V_η, $0 \leqslant \eta \leqslant 1$, denotes the linear process associated to (2.9), which is given by the integral expression (2.11). Hence, by Lemma 4.1 we can apply Theorem 2.20 to get that the processes V_η have exponential dichotomies in $\mathcal{Z}$ with exponents and constants independent of η and if $\{P_\eta(t) : t \in \mathbb{R}\}$ are the associated projections to V_η, see Definition 2.10, for $\mathcal{W} = \mathcal{Z}$ or $\mathcal{Y}$, we have

$$\sup_{t\in\mathbb{R}} \|P_\eta(t) - Q_0(t)\|_{\mathcal{L}(\mathcal{W})} \to 0, \text{ as } \eta \to 0. \tag{6.4}$$

Denote by $\phi_\eta(t) = y(t,\tau;y_0)$ the solution of the initial value problem (2.4). Then, it satisfies

$$\dot{\phi}_\eta = \mathcal{A}_\eta(t)\phi_\eta + h_0(t,\phi_\eta) \tag{6.5}$$

where $\mathcal{A}_\eta(t) = \mathfrak{B}_\eta + D_y f(t, \xi_0^*(t))$ and $h_0(t,\phi_\eta) = f(t,\phi_\eta) - D_y f(t,\xi_0^*(t))\phi_\eta$. But the linear operator $\mathcal{A}_\eta(t)$ generates the linear evolution process $\{V_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ as in (2.11) and by the variations of constants formula we have

$$\phi_\eta(t) = V_\eta(t,\tau)\phi_\eta(\tau) + \int_\tau^t V_\eta(t,s)h_0(s,\phi_\eta(s))ds. \tag{6.6}$$

If we denote by P_η the associated projections to V_η, we project by $P_\eta(t)$ and $I - P_\eta(t)$ and take limits as $\tau \to +\infty$ and $\tau \to -\infty$ respectively, we get

$$P_\eta(t)\phi_\eta(t) = -\int_t^\infty V_\eta(t,s)P_\eta(s)h_0(s,\phi_\eta(s))ds$$

and

$$(I - P_\eta(t))\phi_\eta(t) = \int_{-\infty}^t V_\eta(t,s)(I - P_\eta(s))h_0(s,\phi_\eta(s))ds.$$

Consequently, a unique global bounded solution for (6.6) exists in a small ϵ-neighborhood of ξ_0^*, that is in

$$\mathcal{C}_\epsilon = \{\phi_\eta : \mathbb{R} \to \mathcal{Y} : \sup_{t\in\mathbb{R}} \|\phi_\eta(t) - \xi_0^*(t)\|_{\mathcal{Y}} \leqslant \epsilon\}$$

if and only if

$$\mathcal{T}_\eta(\phi_\eta)(t) = -\int_t^\infty V_\eta(t,s)P_\eta(s)h_0(s,\phi_\eta(s))ds + \int_{-\infty}^t V_\eta(t,s)(I - P_\eta(s))h_0(s,\phi_\eta(s))ds$$

has a unique fixed point in the set $\mathcal{C}_\epsilon$. Let us show that there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \le \epsilon_0$ there exists $\eta(\epsilon) > 0$ such that $\mathcal{T}_\eta$ has a unique fixed point in $\mathcal{C}_\epsilon$ for each $0 \leqslant \eta \leqslant \eta(\epsilon)$. If we show this, then it is clear that in $\mathcal{C}_{\epsilon_0}$ and for all $0 \leqslant \eta \leqslant \eta(\epsilon_0)$ we have one and only one global bounded solution ξ_η^*. As $\eta \to 0$, this solution is the unique fixed point of $\mathcal{T}_\eta$, which will lie in $\mathcal{C}_\epsilon$ for smaller and smaller ϵ. This shows that we have (6.2).

To prove that for all ϵ small, there exists $\eta(\epsilon)$ such that for $0 \leqslant \eta \leqslant \eta(\epsilon)$, $\mathcal{T}_\eta$ takes C_ϵ into itself we recall that

$$\mathcal{T}_0(\xi_0^*)(t) = \xi_0^*(t) = -\int_t^\infty V_0(t,s)P_0(s)h_0(s,\xi_0^*(s))ds + \int_{-\infty}^t V_0(t,s)(I - P_0(s))h_0(s,\xi_0^*(s))ds$$

and note that

$$\begin{aligned}\mathcal{T}_\eta(\phi_\eta)(t) - \xi_0^*(t) = &\int_t^\infty [V_0(t,s)P_0(s)h_0(s,\xi_0^*(s)) - V_\eta(t,s)P_\eta(s)h_0(s,\phi_\eta(s))]ds \\ &+ \int_{-\infty}^t [V_\eta(t,s)(I - P_\eta(s))h_0(s,\phi_\eta(s)) - V_0(t,s)(I - P_0(s))h_0(s,\xi_0^*(s))]ds.\end{aligned}$$

Since both integrals in the above expression are treated in the same manner we will only prove that the first integral can be made smaller than $\frac{\epsilon}{2}$ for suitably small ϵ and η. First note that from Theorem 2.20 and the estimates of Theorem 5.7, there is a $T > 0$ such that

$$\left\| \int_T^\infty [V_0(t,s)P_0(s)h_0(s,\xi_0^*(s)) - V_\eta(t,s)P_\eta(s)h_0(s,\phi_\eta(s))]ds \right\|_{\mathcal{Y}} \leqslant \frac{\epsilon}{4}$$

for all $\eta \in [0,\eta_0]$ and $\phi_\eta \in \mathcal{C}_\epsilon$. Now

$$\begin{aligned}\int_t^T &\|V_0(t,s)P_0(s)h_0(s,\xi_0^*(s)) - V_\eta(t,s)P_\eta(s)h_0(s,\phi_\eta(s))\|_{\mathcal{Y}}ds \\ &\leqslant \int_t^T \|V_0(t,s)P_0(s)[h_0(s,\xi_0^*(s)) - h_0(s,\phi_\eta(s))]\|_{\mathcal{Y}}ds \\ &+ \int_t^T \|V_0(t,s)[P_0(s) - P_\eta(s)]h_0(s,\phi_\eta(s))\|_{\mathcal{Y}}ds \\ &+ \int_t^T \|[V_0(t,s) - V_\eta(t,s)]P_\eta(s)h_0(s,\phi_\eta(s))\|_{\mathcal{Y}}ds.\end{aligned}$$

The last two integrals in the above expression are estimated using the continuity of the projections $\{P_\eta(t) : \ t \in \mathbb{R}\}$ (see Theorem 2.20 and (6.4)) and of the evolution processes $\{V_\eta(t,s) : \ t \geqslant s \in \mathbb{R}\}$ (see Lemma 4.1).

The first integral is estimated using the differentiability of h_0, see (2.6), to show that, given $\delta > 0$, there is an $\epsilon > 0$ such that, if $\|\phi - \xi_0^*\|_{\mathcal{Y}} \leq \epsilon$, then $\|h_0(t,\phi) - h_0(t,\xi_0^*)\|_{\mathcal{Z}} \leqslant \delta\|\phi - \xi_0^*\|_{\mathcal{Y}}$, for all $t \in \mathbb{R}$, as stated in Condition 2.13.

The proof that $\mathcal{T}_\eta$ is a contraction uses the same reasoning used to estimate the first integral (that is, the differentiability of h_0) with an straight forward argument.

This proves the existence of $\xi_\eta^*(\cdot)$, the unique globally defined bounded solution for (2.4) in a small neighborhood of ξ_0^*. Moreover (6.2) holds.

The hyperbolicity of $\xi_\eta^*(\cdot)$ and the statements on the exponential dichotomies, in particular (6.4), follows from the hyperbolicity of $\xi_0^*(\cdot)$, Lemma 4.1. Lemma 4.2 and Theorems 2.20 and 5.7. □

7. Existence of Local Unstable Manifolds as Graphs and Their Continuity

Now we are ready to study the unstable manifolds of a global bounded hyperbolic solution $\xi_\eta^*(\cdot)$ of (2.4), see Definition 2.5.

Note that, writing (2.4) as in (2.15), it suffices to concentrate on the existence of unstable manifolds around the zero solution. That is we consider solutions of

$$\dot{y} = (\mathfrak{B}_\eta + D_y f(t,\xi_\eta^*(t)))y + h_\eta(t,y). \tag{7.1}$$

Since at $(t,0)$ the function h_η is zero with zero derivative, from the continuous differentiability of h_η, uniform with respect to t (see Condition 2.13), we obtain that given $\rho > 0$ there exists $\delta > 0$ such that if $\|y\|_{\mathcal{Y}} < \delta$, then

$$\begin{aligned} &\|h_\eta(t,y)\|_{\mathcal{Z}} \leqslant \rho, \\ &\|h_\eta(t,y) - h_\eta(t,\tilde{y})\|_{\mathcal{Z}} \leqslant \rho\|y - \tilde{y}\|_{\mathcal{Y}}. \end{aligned} \tag{7.2}$$

Remark 7.1. *It is possible to extend h_η outside a ball of radius δ in such a way that the condition (7.2) holds for all $y \in \mathcal{Y}$. Indeed, define $\tilde{h}_\eta : \mathbb{R} \times \mathcal{Y} \to \mathcal{Z}$*

$$\tilde{h}_\eta(t,y) = \begin{cases} h_\eta(t,y), & \|y\|_{\mathcal{Y}} \leqslant \delta \\ h_\eta\left(t, \delta\frac{y}{\|y\|_{\mathcal{Y}}}\right), & \|y\|_{\mathcal{Y}} > \delta. \end{cases}$$

The extension $\tilde{h}_\eta$ becomes globally Lipschitz and its Lipschitz constant is that of h_η restricted to the ball of radius δ.

Under the assumption that (7.2) holds for all $y \in \mathcal{Y}$ with some suitably small $\rho > 0$, in this section we prove that the unstable manifold of a global bounded hyperbolic solution ξ_η^* is given as a graph of a suitable function. Moreover we will also prove the continuity of the global unstable manifolds of ξ_η^*.

From that we will prove the continuity of the local unstable manifolds for the case when h_η only satisfies (7.2) for $\|y\|_{\mathcal{Y}} < \delta$ with $\delta > 0$ suitably small.

Therefore, hereafter we substitute h_η in (7.1) with $\tilde{h}_\eta$ which is assumed to satisfy (7.2) for all y and for suitably small ρ (which will be specified below), that is

$$\dot{y} = (\mathfrak{B}_\eta + D_y f(t,\xi_\eta^*(t)))y + \tilde{h}_\eta(t,y). \tag{7.3}$$

Let $W_\eta^u(0)$ be the global unstable manifold of the solution 0 of (7.3). If $\{Q_\eta(t)\}_{t\in\mathbb{R}}$ are the projections associated with the hyperbolicity of ξ_η^*, we will show that there exists a bounded and Lipschitz continuous function $\Sigma_\eta^{*,u}(t,\cdot) : \mathcal{Y} \to (I - Q_\eta(t))\mathcal{Z} \cap \mathcal{Y}$ such that $\Sigma_\eta^{*,u}(t,y) = \Sigma_\eta^{*,u}(t, Q_\eta(t)(y))$ and

$$W_\eta^u(0) = \{(t,y) : y = Q_\eta(t)y + \Sigma_\eta^{*,u}(t, Q_\eta(t)y),\ y \in \mathcal{Y}\}.$$

The continuity of the unstable manifolds will be derived from the continuity of the functions $\Sigma_\eta^{*,u}(t,\cdot)$ as $\eta \to 0$.

Remark 7.2. *Thus, if $y(t)$ is a solution of (7.3) defined for $t \geq t_0$, we write $y^+(t) = Q_\eta(t)y(t)$ and $y^-(t) = y(t) - y^+(t)$. Then we have*

$$\begin{aligned} y^+(t) &= U_\eta(t,t_0)Q_\eta(t_0)y^+(t_0) + \int_{t_0}^t U_\eta(t,s)Q_\eta(s)\tilde{h}_\eta(s, y^+(s) + y^-(s))ds, \\ y^-(t) &= U_\eta(t,t_0)(I - Q_\eta(t_0))y^-(t_0) + \int_{t_0}^t U_\eta(t,s)(I - Q_\eta(s))\tilde{h}_\eta(s, y^+(s) + y^-(s))ds. \end{aligned} \tag{7.4}$$

Observe that we are looking for a function $\Sigma_\eta^{*,u}(t)$ such that, if $\tau \in \mathbb{R}$ and $\zeta \in R(Q_\eta(\tau))$, then the solution $y(t)$ of (7.3) such that $Q_\eta(\tau)y(\tau) = \zeta$, $(I - Q_\eta(\tau))y(\tau) = \Sigma_\eta^{*,u}(\zeta)$ is such that $y(t)$ is in the graph of $\Sigma_\eta^{*,u}(t,\cdot)$ for all $t < \tau$. This means that, $y^-(t) = \Sigma_\eta^{*,u}(t, y^+(t))$ for all $t < \tau$ and thus (7.4) becomes

$$y^+(t) = U_\eta(t,t_0)Q_\eta(t_0)y^+(t_0) + \int_{t_0}^t U_\eta(t,s)Q_\eta(s)\tilde{h}_\eta(s, y^+(s) + \Sigma_\eta^{*,u}(s, y^+(s)))ds, \tag{7.5}$$

$$y^-(t) = U_\eta(t,t_0)(I - Q_\eta(t_0))y^-(t_0) + \int_{t_0}^t U_\eta(t,s)(I - Q_\eta(s))\tilde{h}_\eta(s, y^+(s) + \Sigma_\eta^{*,u}(s, y^+(s)))ds. \tag{7.6}$$

Also, the solution $y(t)$ should tend to zero as $t \to -\infty$ (in particular, it should stay bounded as $t \to -\infty$). Letting $t_0 \to -\infty$ in (7.6) we have that

$$y^-(t) = \Sigma_\eta^{*,u}(t, y^+(t)) = \int_{-\infty}^t U_\eta(t,s)(I - Q_\eta(s))\tilde{h}_\eta(s, y^+(s) + \Sigma_\eta^{*,u}(s, y^+(s)))ds$$

and, for $t = \tau$,

$$\Sigma_\eta^{*,u}(\tau,\zeta) = y^-(\tau) = \int_{-\infty}^\tau U_\eta(\tau,s)(I - Q_\eta(s))\tilde{h}_\eta(s, y^+(s) + \Sigma_\eta^{*,u}(s, y^+(s)))ds$$

where $y^+ : \mathbb{R} \to \mathcal{Y}$ is the global solution of (7.5) such that $y^+(\tau) = \zeta$.

Note that the equation for y^+ in (7.5) is uncoupled from the equation for y^- and that the last integral equation above can be seen as fixed point problem for $\Sigma_\eta^{*,u}(\tau,\cdot)$.

In order to show existence of the function $\Sigma_\eta^{*,u}(\tau,\cdot)$ we will use the Banach contraction principle. For this, let ω and M be respectively the exponent and constant of the exponential dichotomy of the linearized processes $\{U_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ given by the integral equation (2.10). From Theorem 2.20, ω and M can be taken independent of η for $\eta \in [0,\eta_0]$ and, from Theorem 5.7,

$$\begin{aligned} &\|U_\eta(t,s)(I - Q_\eta(s))z\|_{\mathcal{Y}} \leqslant M(t-s)^{-\gamma}e^{-\omega(t-s)}\|z\|_{\mathcal{Z}}, \quad t > s \\ &\|U_\eta(t,s)Q_\eta(s)z\|_{\mathcal{Y}} \leqslant Me^{\omega(t-s)}\|z\|_{\mathcal{Z}}, \quad t \leqslant s. \end{aligned} \tag{7.7}$$

Now we fix $D > 0$, $L > 0$, $0 < \vartheta < 1$ and choose $\rho > 0$ such that

$$\begin{aligned} \rho < \frac{\omega}{M(1+L)}, \quad & \left[\frac{\rho M}{\omega} + \frac{\rho^2 M^2 (1+L)\Gamma(1-\gamma)}{\omega(2\omega - \rho M(1+L))^{1-\gamma}}\right] \leqslant \frac{1}{2} \\ \frac{\rho M \Gamma(1-\gamma)}{\omega^{1-\gamma}} \leqslant D, \quad & \frac{\rho M}{\omega}(\omega^\gamma \Gamma(1-\gamma) + \frac{L}{M}) \leqslant \vartheta < 1, \\ \frac{\rho M^2(1+L)\Gamma(1-\gamma)}{(\omega - \rho M(1+L))^{1-\gamma}} \leqslant L, \quad & \frac{\rho M}{\omega} + \frac{\rho^2 M^2 (1+L)(1+M)\Gamma(1-\gamma)}{\omega(2\omega - \rho M(1+L))^{1-\gamma}} < \vartheta, \end{aligned} \tag{7.8}$$

where γ is as in (2.5) and (2.12). Note that all quantities above are independent of $0 \leqslant \eta \leqslant \eta_0$.

Then we define the class where we use the Banach contraction principle.

Definition 7.3. *Given $\eta > 0$, denote by $\mathcal{BL}(D, L)$ a complete metric space of all bounded and globally Lipschitz continuous functions $\mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ which satisfy*

$$\begin{aligned} & \Sigma(t,z) = \Sigma(t, Q_\eta(t)z), \ \forall (t,z) \in \mathbb{R} \times \mathcal{Z} \\ & \Sigma(\tau, z) \in (I - Q_\eta(\tau))\mathcal{Z} \cap \mathcal{Y}, \ \forall (t,z) \in \mathbb{R} \times \mathcal{Z} \\ & \sup\{\|\Sigma(\tau, Q_\eta(\tau)z)\|_{\mathcal{Y}}, (\tau,z) \in \mathbb{R} \times \mathcal{Z}\} \leqslant D, \\ & \|\Sigma(\tau, Q_\eta(\tau)z) - \Sigma(\tau, Q_\eta(\tau)\tilde{z})\|_{\mathcal{Y}} \leqslant L \|Q_\eta(\tau)z - Q_\eta(\tau)\tilde{z}\|_{\mathcal{Y}}, \ \forall (\tau, z, \tilde{z}) \in \mathbb{R} \times \mathcal{Z} \times \mathcal{Z}, \end{aligned} \tag{7.9}$$

and the distance of $\Sigma, \tilde{\Sigma} \in \mathcal{BL}(D,L)$ is defined as

$$|||\Sigma(\cdot,\cdot) - \tilde{\Sigma}(\cdot,\cdot)||| := \sup\{\|\Sigma(\tau, Q_\eta(\tau)z) - \tilde{\Sigma}(\tau, Q_\eta(\tau)z)\|_{\mathcal{Y}}, \ (\tau, z) \in \mathbb{R} \times \mathcal{Z}\}.$$

Note that in the definition above we have taken by convenience $\mathcal{Z}$ as the domain and image of the maps $\Sigma \in \mathcal{BL}(D,L)$. However it is clear that $\Sigma(t)$ acts on $Q_\eta(t)\mathcal{Z} \subset \mathcal{Y}$ and the image is in $(I - Q_\eta(t))\mathcal{Z} \cap \mathcal{Y}$ and is a Lipschitz mapping with the metric of $\mathcal{Y}$. Therefore, for $\Sigma \in \mathcal{BL}(D,L)$

$$\{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = Q_\eta(\tau)w + \Sigma(\tau, Q_\eta(\tau)w)\} = \{(\tau, w) \in \mathbb{R} \times \mathcal{Y} : w = Q_\eta(\tau)w + \Sigma(\tau, Q_\eta(\tau)w)\}.$$

Then we have

Theorem 7.4. *Suppose that the above conditions are satisfied. Then, for all $0 \leqslant \eta \leqslant \eta_0$ there exist $\Sigma_\eta^{*,u}(\cdot,\cdot) \in \mathcal{BL}(D,L)$, such that the unstable manifold $W_\eta^u(0)$ of (7.3) is given by*

$$W_\eta^u(0) = \{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = Q_\eta(\tau)w + \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w)\}. \tag{7.10}$$

In addition, if $y(t) = y^+(t) + y^-(t)$, $t \in \mathbb{R}$, is a global solution of (7.3) which is bounded as $t \to -\infty$, then there are constants $\tilde{M} \geqslant 1$ and $\nu > 0$ such that for any $t_0 < t$,

$$\|y^-(t) - \Sigma_\eta^{*,u}(t, y^+(t))\|_{\mathcal{Y}} \leqslant \tilde{M}(t - t_0)^{-\gamma} e^{-\nu(t-t_0)} \|y^-(t_0) - \Sigma_\eta^{*,u}(t, y^+(t_0))\|_{\mathcal{Y}}. \tag{7.11}$$

Proof:
Step 1. For $\tau \in \mathbb{R}$ and arbitrary $\zeta \in Q_\eta(\tau)\mathcal{Z}$, $\Sigma \in \mathcal{LB}(D,L)$ denote by $z^+(t) = \psi(t, \tau, \Sigma)$ the (global) solution of

$$z^+(t) = U_\eta(t,\tau)\zeta + \int_\tau^t U_\eta(t,s)Q_\eta(s)\tilde{h}_\eta(s, z^+(s) + \Sigma(s, z^+(s)))ds, \quad t \in \mathbb{R}. \tag{7.12}$$

Next we define,

$$\Phi(\Sigma)(\tau,\zeta)=\int_{-\infty}^{\tau}U_\eta(\tau,s)(I-Q_\eta(s))\tilde{h}_\eta(s,z^+(s)+\Sigma(s,z^+(s)))ds. \tag{7.13}$$

We will show that, for $\rho>0$ satisfying (7.8), the map Φ takes $\mathcal{BL}(D,L)$ into itself, is a strict contraction, and hence possesses a unique fixed point in $\mathcal{BL}(D,L)$.

First note that, by (7.7), one has

$$\|\Phi(\Sigma)(\tau,\cdot)\|_{\mathcal{Y}}\leqslant\int_{-\infty}^{\tau}\rho M(\tau-s)^{-\gamma}e^{-\omega(\tau-s)}ds=\frac{\rho M\Gamma(1-\gamma)}{\omega^{1-\gamma}}, \tag{7.14}$$

and from (7.8) we have $\sup\{\|\Phi(\Sigma)(\tau,Q_\eta(\tau)z)\|_{\mathcal{Y}},\ (\tau,z)\in\mathbb{R}\times\mathcal{Z}\}\leqslant D$.

Next, suppose that Σ and $\tilde{\Sigma}$ are functions in $\mathcal{BL}(D,L)$, $\zeta,\tilde{\zeta}\in Q_\eta(\tau)\mathcal{Z}$ and denote if $z^+(t)=\psi(t,\tau,\zeta,\Sigma)$, $\tilde{z}^+(t)=\psi(t,\tau,\tilde{\zeta},\tilde{\Sigma})$ as in (7.12). Then

$$\begin{aligned}z^+(t)-\tilde{z}^+(t)&=U_\eta(t,\tau)Q_\eta(\tau)(\zeta-\tilde{\zeta})\\&\quad+\int_{\tau}^{t}U_\eta(t,s)Q_\eta(s)[\tilde{h}_\eta(s,z^+(s)+\Sigma(s,z^+(s)))-\tilde{h}_\eta(s,\tilde{z}^+(s)+\tilde{\Sigma}(s,\tilde{z}^+(s)))]ds,\end{aligned}$$

and with (7.7), (7.2) we obtain, for $t\le\tau$,

$$\begin{aligned}&\|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Y}}\leqslant Me^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}\\&+M\int_t^{\tau}e^{\omega(t-s)}\|\tilde{h}_\eta(s,z^+(s)+\Sigma(s,z^+(s)))-\tilde{h}_\eta(s,\tilde{z}^+(s)+\tilde{\Sigma}(s,\tilde{z}^+(s)))\|_{\mathcal{Z}}ds\\&\leqslant Me^{\omega(t-\tau)}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M|\!|\!|\Sigma-\tilde{\Sigma}|\!|\!|\int_t^{\tau}e^{\omega(t-s)}ds\\&+\rho M(1+L)\int_t^{\tau}e^{\omega(t-s)}\|z^+(s)-\tilde{z}^+(s)\|_{\mathcal{Y}}ds.\end{aligned}$$

If $\phi(t)=e^{-\omega\,(t-\tau)}\|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Y}}$, then, for $t\le\tau$,

$$\phi(t)\leqslant M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M\omega^{-1}|\!|\!|\Sigma-\tilde{\Sigma}|\!|\!|+\rho M(1+L)\int_t^{\tau}\phi(s)ds.$$

By Gronwall's inequality, for $t\le\tau$,

$$\|z^+(t)-\tilde{z}^+(t)\|_{\mathcal{Y}}\leqslant[M\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\rho M\omega^{-1}|\!|\!|\Sigma-\tilde{\Sigma}|\!|\!|]e^{(\omega-\rho M(1+L))(t-\tau)}. \tag{7.15}$$

Thus, from (7.7),

$$\begin{aligned}&\|\Phi(\Sigma)(\tau,\zeta)-\Phi(\tilde{\Sigma})(\tau,\tilde{\zeta})\|_{\mathcal{Y}}\\&\quad\leqslant M\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}\,e^{-\omega(\tau-s)}\|\tilde{h}_\eta(s,z^+(s)+\Sigma(s,z^+(s)))-\tilde{h}_\eta(s,\tilde{z}^+(s)+\tilde{\Sigma}(s,\tilde{z}^+(s)))\|_{\mathcal{Z}}ds\\&\quad\leqslant\rho M\int_{-\infty}^{\tau}(\tau-s)^{-\gamma}e^{-\omega(\tau-s)}\left[(1+L)\|z^+(s)-\tilde{z}^+(s)\|_{\mathcal{Y}}+|\!|\!|\Sigma-\tilde{\Sigma}|\!|\!|\right]ds.\end{aligned}$$

Now, using (7.15), we obtain that

$$\begin{aligned}\|\Phi(\Sigma)(\tau,\zeta)-\Phi(\tilde{\Sigma})(\tau,\tilde{\zeta})\|_{\mathcal{Y}} &\leqslant \frac{\rho M^2(1+L)\Gamma(1-\gamma)}{(\omega-\rho M(1+L))^{1-\gamma}}\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}} \\ &+\frac{\rho M}{\omega}\left[\omega^{\gamma}\Gamma(1-\gamma)+\frac{\rho M(1+L)\Gamma(1-\gamma)}{(\omega-\rho M(1+L))^{1-\gamma}}\right]\|\!|\Sigma-\tilde{\Sigma}\|\!|.\end{aligned} \tag{7.16}$$

Let

$$I_{\Sigma}=\frac{\rho M}{\omega}\left[\omega^{\gamma}\Gamma(1-\gamma)+\frac{\rho M(1+L)\Gamma(1-\gamma)}{(\omega-\rho M(1+L))^{1-\gamma}}\right] \text{ and } I_{\zeta}=\frac{\rho M^2(1+L)\Gamma(1-\gamma)}{(\omega-\rho M(1+L))^{1-\gamma}}.$$

Since, from (7.8), $I_{\Sigma}\leqslant \frac{\rho M}{\omega}(\omega^{\gamma}\Gamma(1-\gamma)+\frac{L}{M})$, it follows from (7.8), (7.16) that

$$\|\Phi(\Sigma)(\tau,\zeta)-\Phi(\tau,\tilde{\Sigma})(\tilde{\zeta})\|_{\mathcal{Y}}\leqslant L\|\zeta-\tilde{\zeta}\|_{\mathcal{Z}}+\vartheta\|\!|\Sigma-\tilde{\Sigma}\|\!|. \tag{7.17}$$

The inequality (7.17) with $\Sigma=\tilde{\Sigma}$ and (7.14) imply that Φ takes $\mathcal{BL}(D,L)$ into $\mathcal{BL}(D,L)$. Due to (7.8), estimate (7.17) with $\zeta=\tilde{\zeta}$ shows that Φ is a contraction map. Therefore, there exists a unique fixed point $\Sigma_{\eta}^{*,u}=\Phi(\Sigma_{\eta}^{*,u})$ in $\mathcal{BL}(D,L)$.

Step 2. Next we prove that

$$W_{\eta}^{u}(0)=\{(\tau,w)\in\mathbb{R}\times\mathcal{Z}: w=Q_{\eta}(\tau)w+\Sigma_{\eta}^{*,u}(\tau,Q_{\eta}(\tau)w)\}. \tag{7.18}$$

To that end assume for a moment that, if $y(t)=y^{+}(t)+y^{-}(t)$, $t\in\mathbb{R}$, is a global solution of (7.3) bounded as $t\to-\infty$, then there are constants $\tilde{M}\geqslant 1$ and $\nu>0$ such that (7.11) holds.

Letting $t_0\to-\infty$ in (7.11) we obtain that $y^{-}(t)=\Sigma_{\eta}^{*,u}(t,y^{+}(t))$ for each $t\in\mathbb{R}$. That also ensures that $\Sigma_{\eta}^{*,u}(t,0)=0$, since 0 is a solution to (7.3). Consequently

$$W_{\eta}^{u}(0)\subset\{(\tau,w)\in\mathbb{R}\times\mathcal{Z}: w=Q_{\eta}(\tau)w+\Sigma_{\eta}^{*,u}(\tau,Q_{\eta}(\tau)w)\}.$$

To prove that $\{(\tau,w)\in\mathbb{R}\times\mathcal{Z}: w=Q_{\eta}(\tau)w+\Sigma_{\eta}^{*,u}(\tau,Q_{\eta}(\tau)w)\}\subset W_{\eta}^{u}(0)$ consider $z_0^{+}\in Q_{\eta}(\tau)\mathcal{Z}$ and $z^{+*}(t)$ satisfying

$$z^{+}(t)=U_{\eta}(t,\tau)Q_{\eta}(\tau)z_0^{+}+\int_{\tau}^{t}U_{\eta}(t,s)Q_{\eta}(s)\tilde{h}_{\eta}(s,z^{+}(s)+\Sigma_{\eta}^{*,u}(s,z^{+}(s)))ds,\ t\in\mathbb{R}. \tag{7.19}$$

This defines a curve $z^{+*}(t)+\Sigma_{\eta}^{*,u}(t,z^{+*}(t))$, $t\in\mathbb{R}$. Recalling (7.13) one can check that

$$\Sigma_{\eta}^{*,u}(t,z^{+*}(t))=\int_{-\infty}^{t}U_{\eta}(t,s)(I-Q_{\eta}(s))\tilde{h}_{\eta}(s,z^{+*}(s)+\Sigma_{\eta}^{*,u}(s,z^{+*}(s))ds,\quad t\in\mathbb{R}. \tag{7.20}$$

Thus $\Sigma_{\eta}^{*,u}(t,z^{+*}(t))$ solves, for $t\geqslant t_0\in\mathbb{R}$,

$$z^{-}(t)=U_{\eta}(t,t_0)(I-Q_{\eta}(t_0))z^{-}(t_0)+\int_{t_0}^{t}U_{\eta}(t,s)(I-Q_{\eta}(s))\tilde{h}_{\eta}(s,z^{+*}(s)+\Sigma_{\eta}^{*,u}(s,z^{+*}(s)))ds,$$

and we conclude that $z^{+*}(t)+\Sigma_{\eta}^{*,u}(t,z^{+*}(t))$, $t\in\mathbb{R}$, is a global solution of (7.3), passing through $z_0^{+}+\Sigma_{\eta}^{*,u}(\tau,z_0^{+})$ at time τ. From (7.20), $\Sigma_{\eta}^{*,u}(t,z^{+*}(t))\to 0$ as $t\to-\infty$. Since $\Sigma_{\eta}^{*,u}(t,0)=0$, the reasoning that lead to (7.15) (with $\tilde{\Sigma}=\Sigma=\Sigma_{\eta}^{*,u}$ and $\tilde{\zeta}=0$) can be used now to ensure that

$$\|z^{+}(t)\|_{\mathcal{Y}}\leqslant M\|z_0^{+}\|_{\mathcal{Y}}e^{(\omega-\rho M(1+L))(t-\tau)}. \tag{7.21}$$

As a consequence $z^+(t) \to 0$ as $t \to -\infty$ and the proof of (7.18) is complete.

Hence, it only remains to prove (7.11). Let $y(t)$ be a solution of (7.3) $\zeta(t) = y^-(t) - \Sigma_\eta^{*,u}(t, y^+(t))$ and $z^+(s,t)$, for $s \leqslant t$, be the solution of

$$z^+(s,t) = U_\eta(s,t)Q_\eta(t)y^+(t) + \int_t^s U_\eta(s,\theta)Q_\eta(\theta)\tilde{h}_\eta(\theta, z^+(\theta,t) + \Sigma_\eta^{*,u}(\theta, z^+(\theta,t)))d\theta, \ s \leqslant t.$$

Hence,

$$\begin{aligned}
&\|z^+(s,t) - y^+(s)\|_{\mathcal{Y}} \\
&\quad = \left\| \int_t^s U_\eta(s,\theta)Q_\eta(\theta)[\tilde{h}_\eta(\theta, z^+(\theta,t) + \Sigma_\eta^{*,u}(\theta, z^+(\theta,t))) - \tilde{h}_\eta(\theta, y^+(\theta) + y^-(\theta))]d\theta \right\|_{\mathcal{Y}} \\
&\quad \leqslant \rho M \int_s^t e^{\omega(s-\theta)} \left[(1+L)\|z^+(\theta,t) - y^+(\theta)\|_{\mathcal{Y}} + \|\zeta(\theta)\|_{\mathcal{Y}}\right] d\theta.
\end{aligned}$$

If $\psi(s) = e^{-\omega s}\|z^+(s,t) - y^+(s)\|_{\mathcal{Y}}$, then

$$\psi(s) \leqslant \rho M(1+L)\int_s^t \psi(\theta)d\theta + \rho M \int_s^t e^{-\omega\theta}\|\zeta(\theta)\|_{\mathcal{Y}}d\theta, \quad s \leqslant t.$$

Using Gronwall's Lemma we have

$$\|z^+(s,t) - y^+(s)\|_{\mathcal{Y}} \leqslant \rho M \int_s^t e^{-(\omega - \rho M(1+L))(\theta - s)}\|\zeta(\theta)\|_{\mathcal{Y}}d\theta, \quad s \leqslant t. \tag{7.22}$$

Now, if $s \leqslant t_0 \leqslant t$, then

$$\begin{aligned}
&\|z^+(s,t) - z^+(s,t_0)\|_{\mathcal{Y}} = \left\|U_\eta(s,t_0)Q_\eta(t_0)[z^+(t_0,t) - y^+(t_0)]\right\|_{\mathcal{Y}} + \\
&\left\|\int_{t_0}^s U_\eta(s,\theta)Q_\eta(\theta)[\tilde{h}_\eta(\theta, z^+(\theta,t) + \Sigma_\eta^{*,u}(\theta, z^+(\theta,t))) - \tilde{h}_\eta(\theta, z^+(\theta,t_0) + \Sigma_\eta^{*,u}(\theta, z^+(\theta,t_0)))]d\theta\right\|_{\mathcal{Y}}
\end{aligned}$$

and using (7.22)

$$\begin{aligned}
\|z^+(s,t) - z^+(s,t_0)\|_{\mathcal{Y}} \leqslant \rho M^2 e^{\omega(s-t_0)} &\int_{t_0}^t e^{-(\omega - \rho M(1+L))(\theta - t_0)}\|\zeta(\theta)\|_{\mathcal{Y}}d\theta \\
&+ \rho M \int_s^{t_0} e^{\omega(s-\theta)}(1+L)\|z^+(\theta,t) - z^+(\theta,t_0)\|_{\mathcal{Y}}d\theta.
\end{aligned}$$

From Gronwall's lemma it follows that, for $s \leqslant t_0 \leqslant t$,

$$\|z^+(s,t) - z^+(s,t_0)\|_{\mathcal{Y}} \leqslant \rho M^2 \int_{t_0}^t e^{-(\omega - \rho M(1+L))(\theta - s)}\|\zeta(\theta)\|_{\mathcal{Y}}d\theta. \tag{7.23}$$

We use this to estimate $\zeta(t)$. Note that

$$
\begin{aligned}
&\zeta(t) - U_\eta(t,t_0)(I-Q_\eta(t_0))\zeta(t_0)\\
&= y^-(t) - \Sigma_\eta^{*,u}(t,y^+(t)) - U_\eta(t,t_0)(I-Q_\eta(t_0))[y^-(t_0) - \Sigma_\eta^{*,u}(t_0,y^+(t_0))]\\
&= \int_{t_0}^t U_\eta(t,s)(I-Q_\eta(s))\tilde{h}_\eta(s,y^+(s)+y^-(s))ds\\
&- \Sigma_\eta^{*,u}(t,y^+(t)) + U_\eta(t,t_0)(I-Q_\eta(t_0))\Sigma_\eta^{*,u}(t_0,y^+(t_0))\\
&= \int_{t_0}^t U_\eta(t,s)(I-Q_\eta(s))[\tilde{h}_\eta(s,y^+(s)+y^-(s)) - \tilde{h}_\eta(s,z^+(s,t)+\Sigma_\eta^{*,u}(s,z^+(s,t)))]ds\\
&-\int_{-\infty}^{t_0} U_\eta(t,s)(I-Q_\eta(s))[\tilde{h}_\eta(s,z^+(s,t)+\Sigma_\eta^{*,u}(s,z^+(s,t)))-\tilde{h}_\eta(s,z^+(s,t_0)+\Sigma_\eta^{*,u}(s,z^+(s,t_0)))]ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\zeta(t)-U_\eta(t,t_0)(I-Q_\eta(t_0))\zeta(t_0)\|_{\mathcal{Y}}\\
&\qquad \leqslant \rho M \int_{t_0}^t (t-s)^{-\gamma} e^{-\omega(t-s)} \left[\|y^+(s)-z^+(s,t)\|_{\mathcal{Y}} + \|y^-(s)) - \Sigma_\eta^{*,u}(s,z^+(s,t))\|_{\mathcal{Y}}\right] ds\\
&\qquad + \rho M(1+L) \int_{-\infty}^{t_0} (t-s)^{-\gamma} e^{-\omega(t-s)} \|z^+(s,t)-z^+(s,t_0)\|_{\mathcal{Y}} ds
\end{aligned}
$$

and, using (7.22) and (7.23), we obtain

$$
\begin{aligned}
\|\zeta(t) - U_\eta(t,t_0)(I-Q_\eta(t_0))\zeta(t_0)\|_{\mathcal{Y}} &\leqslant \rho M \int_{t_0}^t (t-s)^{-\gamma} e^{-\omega(t-s)} \|\zeta(s)\|_{\mathcal{Y}} ds\\
&+ \rho^2 M^2 (1+L) \int_{t_0}^t (t-s)^{-\gamma} e^{-\omega(t-s)} \int_s^t e^{-(\omega-\rho M(1+L))(\theta-s)} \|\zeta(\theta)\|_{\mathcal{Y}} d\theta\, ds\\
&+ \rho^2 M^3 (1+L) \int_{-\infty}^{t_0} (t-s)^{-\gamma} e^{-\omega(t-s)} \int_{t_0}^t e^{-(\omega-\rho M(1+L))(\theta-s)} \|\zeta(\theta)\|_{\mathcal{Y}} d\theta\, ds,
\end{aligned}
$$

so that

$$
\begin{aligned}
\|\zeta(t) - U_\eta(t,t_0)(I-Q_\eta(t_0))\zeta(t_0)\|_{\mathcal{Y}} &\leqslant \rho M \int_{t_0}^{t} (t-s)^{-\gamma} e^{-\omega(t-s)} \|\zeta(s)\|_{\mathcal{Y}} ds \\
&+ \rho^2 M^2 (1+L) \int_{t_0}^{t} e^{-\omega(t-\theta)} \|\zeta(\theta)\|_{\mathcal{Y}} \int_{t_0}^{\theta} (t-s)^{-\gamma} e^{-(2\omega-\rho M(1+L))(\theta-s)} ds\, d\theta \\
&+ \rho^2 M^3 (1+L) \int_{t_0}^{t} e^{-\omega(t-\theta)} \|\zeta(\theta)\|_{\mathcal{Y}} \int_{-\infty}^{t_0} (t-s)^{-\gamma} e^{-(2\omega-\rho M(1+L))(\theta-s)} ds\, d\theta \\
&\leqslant \rho M \int_{t_0}^{t} (t-s)^{-\gamma} e^{-\omega(t-s)} \|\zeta(s)\|_{\mathcal{Y}} ds \\
&+ \rho^2 M^2 (1+L) \int_{t_0}^{t} (t-\theta)^{-\gamma} e^{-\omega(t-\theta)} \|\zeta(\theta)\|_{\mathcal{Y}} \int_{t_0}^{\theta} e^{-(2\omega-\rho M(1+L))(\theta-s))} ds\, d\theta \\
&+ \rho^2 M^3 (1+L) \int_{t_0}^{t} (t-\theta)^{-\gamma} e^{-\omega(t-t_0+t_0-\theta)} \|\zeta(\theta)\|_{\mathcal{Y}} \int_{-\infty}^{t_0} e^{-(2\omega-\rho M(1+L))(\theta-t_0+t_0-s))} ds\, d\theta \\
&\leqslant \left[\rho M + \frac{\rho^2 M^2 (1+L)}{2\omega - \rho M(1+L))}\right] e^{-\omega(t-t_0)} \int_{t_0}^{t} (t-s)^{-\gamma} e^{\omega(s-t_0)} \|\zeta(s)\|_{\mathcal{Y}} ds \\
&+ \frac{\rho^2 M^3 (1+L)}{2\omega - \rho M(1+L)} e^{-\omega(t-t_0)} \int_{t_0}^{t} (t-\theta)^{-\gamma} e^{-(2\omega-\rho M(1+L))(\theta-t_0)} e^{\omega(\theta-t_0)} \|\zeta(\theta)\|_{\mathcal{Y}} d\theta \\
&\leqslant \left[\rho M + \frac{\rho^2 M^2 (1+L)(1+M)}{2\omega - \rho M(1+L)}\right] e^{-\omega(t-t_0)} \int_{t_0}^{t} (t-s)^{-\gamma} e^{\omega(s-t_0)} \|\zeta(s)\|_{\mathcal{Y}} ds
\end{aligned}
$$

and therefore

$$
e^{\omega(t-t_0)} \|\zeta(t)\|_{\mathcal{Y}} \leqslant M \|\zeta(t_0)\|_{\mathcal{Y}} + \left[\rho M + \frac{\rho^2 M^2 (1+L)(1+M)}{2\omega - \rho M(1+L)}\right] \int_{t_0}^{t} (t-s)^{-\gamma} e^{\omega(s-t_0)} \|\zeta(s)\|_{\mathcal{Y}} ds.
$$

Now the singular Gronwall's lemma we have that there is a constant K depending only on γ such that

$$
\|\zeta(t)\|_{\mathcal{Y}} \leqslant K M_1 \|\zeta(t_0)\|_{\mathcal{Y}} (t-t_0)^{-\gamma} e^{-\nu(t-t_0)}, \tag{7.24}
$$

where

$$
\nu = \omega - \left[\left(\rho M + \frac{\rho^2 M^2 (1+L)(1+M)}{2\omega - \rho M(1+L)}\right) \Gamma(1-\gamma)\right]^{\frac{1}{1-\gamma}}.
$$

This proves (7.11) and concludes the proof of the theorem.□

7.1. **Continuity of Unstable Manifolds.** In this section we prove the continuity of unstable manifolds with respect to the parameter η.

Remember that we can decompose a solution $y_\eta(t)$ of (7.3) as $y_\eta^+(t) = Q_\eta(t)y_\eta(t)$ and $y_\eta^-(t) = (I - Q_\eta(t))y_\eta(t)$. Then

$$\begin{aligned} y_\eta^+(t) &= U_\eta(t,t_0)Q_\eta(t_0)y_\eta^+(t_0) + \int_{t_0}^t U_\eta(t,s)Q_\eta(s)\tilde{h}_\eta(s, y_\eta^+(s) + y_\eta^-(s))ds, \\ y_\eta^-(t) &= U_\eta(t,t_0)(I - Q_\eta(t_0))y_\eta^-(t_0) + \int_{t_0}^t U_\eta(t,s)(I - Q_\eta(s))\tilde{h}_\eta(s, y_\eta^+(s) + y_\eta^-(s))ds. \end{aligned} \tag{7.25}$$

Let $D > 0$, $L > 0$, $0 < \vartheta < 1$, $\rho > 0$ be such that (7.8) is satisfied and assume (7.2).

The continuity of the unstable manifolds is stated in the following result.

Theorem 7.5. *Suppose that the above conditions are satisfied and* (7.8) *holds, so that for* $0 \leqslant \eta \leqslant \eta_0$ *there exists a function* $\Sigma_\eta^{*,u} \in \mathcal{BL}(D, L)$*, such that the unstable manifold* $W_\eta^u(0)$ *of the solution* 0 *to* (7.3) *is given by*

$$W_\eta^u(0) = \{(\tau, w) \in \mathbb{R} \times \mathcal{Z} : w = Q_\eta(\tau)w + \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w)\};$$

then, from Theorem 7.4, for every $\zeta \in Q_\eta(\tau)\mathcal{Z}$,

$$\Sigma_\eta^{*,u}(\tau, \zeta) = \int_{-\infty}^\tau U_\eta(\tau, s)(I - Q_\eta(s))\tilde{h}_\eta(s, z^+(s) + \Sigma_\eta^{*,u}(s, z^+(s)))ds. \tag{7.26}$$

In addition, for each $r > 0$ *and* $\tau \in \mathbb{R}$,

$$\sup_{t \leqslant \tau} \sup_{\substack{z \in \mathcal{Y} \\ \|z\|_\mathcal{Y} \leqslant r}} \left\{ \|Q_\eta(t)z - Q_0(t)z\|_\mathcal{Y} + \|\Sigma_\eta^{*,u}(t, Q_\eta(t)z) - \Sigma_0^{*,u}(t, Q_0(t)z)\|_\mathcal{Y} \right\} \xrightarrow{\eta \to 0} 0.$$

Proof: Note that

$$\begin{aligned} \|\Sigma_\eta^{*,u}(t, Q_\eta(t)z) - \Sigma_0^{*,u}(t, Q_0(t)z)\|_\mathcal{Y} &\leq \|\Sigma_\eta^{*,u}(t, Q_\eta(t)Q_0(t)z) - \Sigma_0^{*,u}(t, Q_0(t)z)\|_\mathcal{Y} \\ &+ \|\Sigma_\eta^{*,u}(t, Q_\eta(t)Q_0(t)z + Q_\eta(t)(I - Q_0(t))z) - \Sigma_\eta^{*,u}(t, Q_\eta(t)Q_0(t)z)\|_\mathcal{Y}. \end{aligned}$$

Using that $\Sigma_\eta^{*,u}$ is Lipschitz continuous (uniformly with respect to η) and the continuity of projections (see Theorem 5.4) we conclude that the last term in the above expression converges to zero as $\eta \to 0$ uniformly for $t \in \mathbb{R}$. Hence, we only have to prove that, given $r > 0$ and $\tau \in \mathbb{R}$,

$$\sup_{t \leqslant \tau} \sup_{\substack{z \in R(Q_0(\tau)) \\ \|z\|_\mathcal{Y} \leqslant r}} \|\Sigma_\eta^{*,u}(t, Q_\eta(t)Q_0(t)z) - \Sigma_0^{*,u}(t, Q_0(t)z)\|_\mathcal{Y} \xrightarrow{\eta \to 0} 0. \tag{7.27}$$

Now, if $z \in Q_0(\tau)\mathcal{Z}$ with $\|z\|_\mathcal{Y} \leqslant r$, for any $\eta \in [0, 1]$, as in (7.19), we have

$$z_\eta^+(t) = U_\eta(t, \tau)Q_\eta(\tau)z + \int_\tau^t U_\eta(t,s)Q_\eta(s)\tilde{h}_\eta(s, z_\eta^+(s) + \Sigma_\eta^{*,u}(s, z_\eta^+(s)))ds, \; t \in \mathbb{R}. \tag{7.28}$$

Using (7.26) we have that, for $t \leqslant \tau$,

$$\begin{aligned}
&\Sigma_\eta^{*,u}(t, Q_\eta(t)z_0^+(t)) - \Sigma_0^{*,u}(t, Q_0(t)z_0^+(t)) \\
&= \int_{-\infty}^t [U_\eta(t,s)(I - Q_\eta(s)) - U_0(t,s)(I - Q_0(s))]\tilde{h}_\eta(s, z_\eta^+ + \Sigma_\eta^{*,u}(s, z_\eta^+))ds \\
&- \int_{-\infty}^t U_0(t,s)(I - Q_0(s))[\tilde{h}_0(z_0^+ + \Sigma_0^{*,u}(s, z_0^+)) - \tilde{h}_\eta(s, z_0^+ + \Sigma_0^{*,u}(s, z_0^+))]ds \\
&- \int_{-\infty}^t U_0(t,s)(I - Q_0(s))[\tilde{h}_\eta(s, z_0^+ + \Sigma_0^{*,u}(s, z_0^+)) - \tilde{h}_\eta(s, z_\eta^+ + \Sigma_\eta^{*,u}(s, z_\eta^+))]ds \\
&=: I_1(\eta)(t) + I_2(\eta)(t) + I_3(\eta)(t).
\end{aligned} \tag{7.29}$$

Using the continuity of the processes U_η (see Lemma 4.2), their uniform (w.r.t η) dichotomy and the continuity of the projections (see Theorem 6.1), we obtain that $I_1(\eta)(t) \to 0$ in $\mathcal{Y}$ as $\eta \to 0$ uniformly for $t \in \mathbb{R}$. Also, $I_2(\eta)(t) \to 0$ in $\mathcal{Y}$ as $\eta \to 0$ uniformly for $t \in \mathbb{R}$ from the convergence of ξ_η^* to ξ_0^*.

Next let us estimate $I_3(\eta)$. Note that it is for this term only that we can not obtain uniform in time convergence to 0, but only uniformly for $t \leqslant \tau$. From (7.28), proceeding in as in case of (7.15) one can get the analogous expression to (7.21) (with constants independent of η)

$$\|z_\eta^+(t)\|_{\mathcal{Y}} \leqslant M e^{(\omega - \rho M(1+L))(t-\tau)} \|Q_\eta(\tau)z\|_{\mathcal{Y}}, \quad t \leqslant \tau \tag{7.30}$$

where ω and M are the exponent and constant of the exponential dichotomy of $U_\eta(t,s)$.

Note that, for $t \leqslant \tau$,

$$\begin{aligned}
\|I_3(\eta)(t)\|_{\mathcal{Y}} &\leqslant \rho M \int_{-\infty}^t (t-s)^{-\gamma} e^{-\omega(t-s)} \left[\|z_\eta^+(s) - z_0^+(s)\|_{\mathcal{Y}} + \|\Sigma_\eta^{*,u}(s, z_\eta^+(s)) - \Sigma_0^{*,u}(s, z_0^+(s))\|_{\mathcal{Y}}\right] ds \\
&\leqslant \rho M(1+L) \int_{-\infty}^t (t-s)^{-\gamma} e^{-\omega(t-s)} \|z_\eta^+(s) - z_0^+(s)\|_{\mathcal{Y}} ds + \frac{\rho M \Gamma(1-\gamma)}{\omega^{1-\gamma}} |\!|\!| \Sigma_\eta^{*,u} - \Sigma_0^{*,u} |\!|\!|_{r,\tau},
\end{aligned}$$

with

$$|\!|\!| \Sigma_\eta^{*,u} - \Sigma_0^{*,u} |\!|\!|_{r,\tau} = \sup_{s \leqslant \tau} \sup_{\substack{z \in R(Q_0(\tau)) \\ \|z\|_{\mathcal{Y}} \leqslant r}} \|\Sigma_\eta^{*,u}(s, Q_\eta(s)z_0^+(s)) - \Sigma_0^{*,u}(s, z_0^+(s))\|_{\mathcal{Y}} \tag{7.31}$$

where $z_0^+(\cdot)$ is defined in (7.28) with $\eta = 0$.

Hereafter we will use the notation $o(1)$ with the meaning that the expression represented by it goes to zero as $\eta \to 0$. Replacing this in (7.29) we have, for $z_0^+(t)$ as in (7.28) with $\eta = 0$,

$$\begin{aligned}
\|\Sigma_\eta^{*,u}(t, Q_\eta(t)z_0^+(t)) - \Sigma_0^{*,u}(t, Q_0(t)z_0^+(t))\|_{\mathcal{Y}} &\leqslant o(1) + \frac{\rho M \Gamma(1-\gamma)}{\omega^{1-\gamma}} |\!|\!| \Sigma_\eta^{*,u} - \Sigma_0^{*,u} |\!|\!|_{r,\tau} \\
&\quad + \rho M(1+L) \int_{-\infty}^t (t-s)^{-\gamma} e^{-\omega(t-s)} \|z_\eta^+(s) - z_0^+(s)\|_{\mathcal{Y}} ds.
\end{aligned} \tag{7.32}$$

Now, from (7.28) we have that, for $t \leqslant \tau$,

$$
\begin{aligned}
&\|z_\eta^+(t) - z_0^+(t)\|_{\mathcal{Y}} \leqslant \big\|U_\eta(t,\tau)Q_\eta(\tau)z - U_0(t,\tau)Q_0(\tau)z\big\|_{\mathcal{Y}} \\
&+\Big\|\int_\tau^t [U_\eta(t,s)Q_\eta(s)h_\eta(s, z_\eta^+ + \Sigma_\eta^{*,u}(s,z_\eta^+))ds - U_0(t,s)Q_0(s)h_0(s,z_0^+ + \Sigma_0^{*,u}(s,z_0^+))]ds\Big\|_{\mathcal{Y}} \\
&\leqslant \big\|U_\eta(t,\tau)Q_\eta(\tau)z - U_0(t,\tau)Q_0(\tau)z\big\|_{\mathcal{Y}} \\
&+\Big\|\int_\tau^t [U_\eta(t,s)Q_\eta(s) - U_0(t,s)Q_0(s)]h_0(s,z_0^+ + \Sigma_0^{*,u}(z_0^+))ds\Big\|_{\mathcal{Y}} \\
&+\Big\|\int_\tau^t U_\eta(t,s)Q_\eta(s)\left[h_\eta(s,z_0^+(s) + \Sigma_0^{*,u}(z_0^+(s))) - h_0(s,z_0^+(s) + \Sigma_0^{*,u}(z_0^+(s)))\right]ds\Big\|_{\mathcal{Y}} \\
&+\Big\|\int_\tau^t U_\eta(t,s)Q_\eta(s)\left[h_\eta(s,z_\eta^+(s) + \Sigma_\eta^{*,u}(s,z_\eta^+(s))) - h_\eta(s,z_0^+(s) + \Sigma_0^{*,u}(s,z_0^+(s)))\right]ds\Big\|_{\mathcal{Y}}.
\end{aligned}
$$

Using that the first three terms above remain bounded, we write

$$
\begin{aligned}
&\|z_\eta^+(t) - z_0^+(t)\|_{\mathcal{Y}} \leqslant \\
&\leqslant o(1)+\rho M\int_t^\tau e^{\omega(t-s)}[(1+L)\|z_\eta^+ - z_0^+\|_{\mathcal{Y}} + \|\Sigma_\eta^{*,u}(s,Q_\eta(s)z_0^+) - \Sigma_0^{*,u}(s,z_0^+)\|_{\mathcal{Y}}]ds \\
&\leqslant o(1)+\frac{\rho M}{\omega}|\!|\!|\Sigma_\eta^{*,u} - \Sigma_0^{*,u}|\!|\!|_{r,\tau} + \rho M(1+L)\int_t^\tau e^{\omega(t-s)}\|z_\eta^+ - z_0^+\|_{\mathcal{Y}}ds
\end{aligned}
$$

and, from Gronwall's lemma,

$$
\|z^+(t) - z_0^+(t)\|_{\mathcal{Y}} \leqslant \big(o(1) + \frac{\rho M}{\omega}|\!|\!|\Sigma_\eta^{*,u} - \Sigma_0^{*,u}|\!|\!|_{r,\tau}\big)e^{(\omega-\rho M(1+L))(t-\tau)},\ t\leqslant\tau. \tag{7.33}
$$

Applying (7.33) to (7.32) we obtain that

$$
\begin{aligned}
\|\Sigma_\eta^{*,u}(t,&Q_\eta(t)z_0^+(t)) - \Sigma_0^{*,u}(Q_0(t)z_0^+(t))\|_{\mathcal{Y}} \leqslant o(1) + \frac{\rho M}{\omega}|\!|\!|\Sigma_\eta^{*,u} - \Sigma_0^{*,u}|\!|\!|_{r,\tau} \\
&+\rho M(1+L)\int_{-\infty}^t (t-s)^{-\gamma}e^{-(2\omega-\rho M(1+L))(t-s)}\left[o(1) + \frac{\rho M}{\omega}|\!|\!|\Sigma_\eta^{*,u} - \Sigma_0^{*,u}|\!|\!|_{r,\tau}\right]ds \\
&\leqslant o(1) + \left[\frac{\rho M}{\omega} + \frac{\rho^2M^2(1+L)\Gamma(1-\gamma)}{\omega(2\omega-\rho M(1+L))^{1-\gamma}}\right]|\!|\!|\Sigma_\eta^{*,u} - \Sigma_0^{*,u}|\!|\!|_{r,\tau} \\
&=: o(1) + \frac{1}{2}|\!|\!|\Sigma_\eta^{*,u} - \Sigma_0^{*,u}|\!|\!|_{r,\tau}.
\end{aligned} \tag{7.34}
$$

It follows from (7.34) that $|\!|\!|\Sigma_\eta^{*,u} - \Sigma_0^{*,u}|\!|\!|_{r,\tau} \leqslant o(1)$, which completes the proof. □

7.2. **Local unstable manifolds as graphs and their continuity.** In this section we use the results in Section 7.1 to obtain the existence and continuity of local unstable manifolds when h, h_η only satisfy (7.2) for $\|z\|_{\mathcal{Y}} < \delta$ with $\delta > 0$ suitably small.

Note that in view of (7.1), (7.3) and Remark 7.1 it is enough to prove that some solutions on the stable manifold of (7.3) are sufficiently small.

Proof Theorem 2.19: According to Remark 7.1 and Theorem 7.4, we only need to ensure that, given $\delta > 0$ there is $0 < \delta' \leqslant \delta$ such that any solution $z^+(t) + \Sigma^*_\eta(t, z^+(t))$ on the unstable manifold of (7.3) which satisfies $\|z^+(t_0) + \Sigma^*_\eta(t_0, z^+(t_0))\|_\mathcal{Y} < \delta'$, for some $t_0 \in \mathbb{R}$, satisfies $\|z^+(t) + \Sigma^*_\eta(t, z^+(t))\|_\mathcal{Y} < \delta$, for all $t \leqslant t_0$. Since, from (7.19), $z^+(t)$ is the solution of

$$z^+_\eta(t) = U_\eta(t, t_0)Q_\eta(t_0)z^+_\eta(t_0) + \int_{t_0}^t U_\eta(t, s)Q_\eta(s)\tilde{h}_\eta(s, z^+_\eta(s) + \Sigma^{*u}_\eta(s, z^+_\eta(s)))ds, \quad t \leqslant t_0,$$

we have, from (7.21), that, for $t \leqslant t_0$,

$$\|z^+(t)\|_\mathcal{Y} \leqslant Me^{(\omega - \rho M(1+L))(t-t_0)}\|z^+(t_0)\|_\mathcal{Y}. \tag{7.35}$$

Thus, since Σ^{*u}_η is Lipschitz continuous (uniform w.r.t η) and $\Sigma^{*u}_\eta(t, 0) = 0$, we have that

$$\|\Sigma^{*,u}_\eta(t, z^+(t))\|_\mathcal{Y} \leqslant MLe^{(\omega - \rho M(1+L))(t-t_0)}\|z^+(t_0)\|_\mathcal{Y}, \quad t \leqslant t_0$$

and the proof now follows easily. □

8. **Continuity of global dynamical structures**

The results outlined in Section 2 are the basic "tools" we use to obtain continuity of the asymptotic dynamics of the system. Therefore, we need to figure out a way to translate this information to obtain the continuity of more global dynamical structures. Let us start defining precisely what we mean by upper and lower semicontinuity of sets,

Definition 8.1. *A family $\{A_\eta : \eta \in [0, 1]\}$ of subsets of $\mathcal{Y}$ is upper semicontinuous at $\eta = 0$ if* $\mathrm{dist}_\mathcal{Y}(A_\eta, A_0) \overset{\eta\to 0}{\longrightarrow} 0$ *and lower semicontinuous at $\eta = 0$ if* $\mathrm{dist}_\mathcal{Y}(A_0, A_\eta) \overset{\eta\to 0}{\longrightarrow} 0$*, where* dist *is defined in (2.14).*

A family of time dependent sets of $\mathcal{Y}$, $\mathcal{A}_\eta = \{A_\eta(t),\ t \in \mathbb{R}\}$, $\eta \in [0, 1]$, is upper (resp. lower) semicontinuous at $\eta = 0$ if it is upper (resp. lower) semicontinuous for each fixed $t \in \mathbb{R}$.

One immediate result in this direction is the lower semicontinuity of "global unstable manifolds" of hyperbolic global bounded solutions.

Corollary 8.2. *We have the following*
i) If $\xi^*_0(\cdot)$ and $\xi^*_\eta(\cdot)$ are the solutions from Theorem 2.18 then for each $t \in \mathbb{R}$ and each $\gamma_0 \in W^u(\xi^*_0)(t)$ there exists $\gamma_\eta \in W^u(\xi^*_\eta)(t)$ with $\gamma_\eta \to \gamma_0$ in $\mathcal{Y}$ as $\eta \to 0$. As a consequence, $W^u(\xi^*_\eta)(t)$ is lower semicontinuous at $\eta = 0$.

We also have

$$\sup_{t\in\mathbb{R}}\left\{\mathrm{dist}_Y\left(\bigcup_{0\leqslant\tau\leq T} T_0(t+\tau, t)(W^u_{loc}(\xi^*_0, \rho)(t)), \bigcup_{0\leqslant\tau\leq T} T_\eta(t+\tau, t)(W^u_{loc}(\xi^*_\eta, \rho)(t))\right)\right\} \overset{\eta\to 0}{\longrightarrow} 0. \tag{8.1}$$

Moreover, if $W^u(\xi^*_0)(t)$ is a relatively compact set in $\mathcal{Y}$ then

$$\mathrm{dist}_\mathcal{Y}(\overline{W^u(\xi^*_0)(t)}, W^u(\xi^*_\eta)(t)) \overset{\eta\to 0}{\longrightarrow} 0, \qquad \text{for all } t \in \mathbb{R}. \tag{8.2}$$

ii) If we denote by $\{\xi^*_{i,0}(\cdot)\}_{i\in I_0}$ and $\{\xi^*_{i,\eta}(\cdot)\}_{i\in I_\eta}$ the set of hyperbolic global bounded solutions of equations (2.3) and (2.4), respectively and if $\cup_{i\in I_0} W^u(\xi^*_{i,0})(t)$ is a relatively compact set of $\mathcal{Y}$, then

$$\operatorname{dist}_{\mathcal{Y}}\left(\overline{\bigcup_{i\in I_0} W^u(\xi^*_{i,0})(t)},\ \bigcup_{i\in I_\eta} W^u(\xi^*_{i,\eta})(t)\right) \stackrel{\eta\to 0}{\longrightarrow} 0, \quad \textit{for all } t\in\mathbb{R}. \tag{8.3}$$

Proof. The proof goes as follows.
i) If $\gamma_0 \in W^u(\xi^*_0)(t)$, from Remark 2.7 ii) we have that there exists $s_0 \le t$ and an element $\psi_0 \in W^u_{loc}(\xi^*_0,\rho)(s_0)$ such that $\gamma_0 = T(t,s_0)(\psi_0)$. From Theorem 2.19 there exists a sequence of $\psi_\eta \in W^u_{loc}(\xi^*_\eta,\rho)(s_0)$ such that $\psi_\eta \to \psi_0$ as $\eta\to 0$. From Proposition 2.17 we have $\gamma_\eta \equiv T_\eta(t,s_0)(\psi_\eta) \to T_0(t,s_0)(\psi_0) \equiv \gamma_0$ as $\eta \to 0$ and from Remark 2.7 we have $\gamma_\eta \in W^u(\xi_\eta)(t)$. This shows the first part of i). Now, from Lemma 2.7 in [17] we have the lower semicontinuity property.

To show (8.1) we use Theorem 2.19 and Proposition 2.17.

To show (8.2) we use the precompactness of $W^u(\xi^*_0)(t)$. Observe that if $W^u(\xi^*_0)(t)$ is a relatively compact set in $\mathcal{Y}$ then, for each $\epsilon > 0$ small, there exists a finite number of points $\gamma^1_0, ..\gamma^N_0 \in W^u(\xi^*_0)(t)$ such that $\overline{W^u(\xi^*_0)(t)} \subset \cup_{i=1}^N B_{\mathcal{Y}}(\gamma^i_0,\epsilon)$. For each of this γ^i_0 there exists $\gamma^i_\eta \in W^u(\xi^*_\eta)(t)$ with $\gamma^i_\eta \to \gamma^i_0$ in $\mathcal{Y}$. Hence, we can choose $\eta_0 = \eta_0(\epsilon) > 0$ small such that for $0<\eta<\eta_0$, $\|\gamma^i_0 - \gamma^i_\eta\|_{\mathcal{Y}} \le \epsilon$. From here we have that $\operatorname{dist}_{\mathcal{Y}}(\gamma^i_0, W^u(\xi^*_\eta)(t)) \le \epsilon$ for $0<\eta<\eta_0$ which implies $\operatorname{dist}_{\mathcal{Y}}(\overline{W^u(\xi^*_0)(t)}, W^u(\xi^*_\eta)(t)) \le 2\epsilon$. This shows (8.2).
ii) This proof follows directly from i). □

We are interested in obtaining continuity results for other global dynamical objects like the "attractor" of the system. We define the notion of pullback attractor, which is a suitable concept of attractors for the non-autonomous problems we consider in this paper.

Definition 8.3. *Let $\{S(t,\tau):\ t\geqslant\tau\}$ be an evolution process and $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ a time dependent family of sets in $\mathcal{Y}$. We say that $\mathcal{A}$ pullback* attracts *a bounded set $B\subset\mathcal{Y}$ under $\{S(t,\tau):\ t\geqslant\tau\}$ if*

$$\lim_{\tau\to-\infty} \operatorname{dist}_{\mathcal{Y}}(S(t,\tau)B, A(t)) = 0,\ \forall t\in\mathbb{R}.$$

We say that $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ is the pullback attractor *for the process $\{S(t,\tau):\ t\geqslant\tau\}$ if it is invariant, $\cup_{t\in\mathbb{R}}A(t)$ is bounded, $A(t)$ is compact for each $t\in\mathbb{R}$ and $\mathcal{A}$ pullback attracts each bounded set $B\subset\mathcal{Y}$.*

Remark 8.4.
i) Observe that if $S(t,\tau)$ is the evolution process associated with a nonlinear semigroup, then the concept of pullback attractor coincides with the classical one of global attractor (see [24, 38, 7, 31, 36]).
ii) Note that the pullback attraction relies on the property that solutions that started long away in the past time, approach some set of states at each present time t.

Hence, let us first assume that the evolution processes associated to (2.3) and (2.4)

$$\{T_\eta(t,\tau) : t\geqslant\tau\in\mathbb{R}\} \text{ has a } \textit{pullback attractor } \mathcal{A}_\eta = \{A_\eta(t) : t\in\mathbb{R}\} \tag{8.4}$$

for each $0\leqslant\eta\leqslant 1$.

We can show an upper semicontinuity result for the attractors (cf. [16, 17]),

Proposition 8.5. *Consider the family $\{T_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, of nonlinear evolution processes and assume that* (2.13) *is satisfied. Suppose that, for each $\eta \in [0,1]$ the nonlinear process $\{T_\eta(t,\tau) : t \geq \tau\}$ has a global pullback attractor $\mathcal{A}_\eta = \{A_\eta(t)\}_{t\in\mathbb{R}}$, satisfying that*

$$\bigcup_{t\in\mathbb{R}} \bigcup_{\eta\in[0,\eta_0]} A_\eta(t) \quad \textit{is bounded.} \tag{8.5}$$

Then, the family of pullback attractors is upper semicontinuous at $\eta = 0$. More precisely, we have

$$\mathrm{dist}_{\mathcal{Y}}(A_\eta(t), A_0(t)) \overset{\eta\to 0}{\longrightarrow} 0, \qquad \textit{for all } t \in \mathbb{R} \tag{8.6}$$

Proof. Let B be a bounded set in $\mathcal{Y}$ such that $\overline{\bigcup_{t\in\mathbb{R}} \bigcup_{\eta\in[0,\eta_0]} A_\eta(t)} \subset B$ and let $\delta > 0$ be a positive small number.

From the pullback attraction properties of $\mathcal{A}_0$ we have that there exists a $T > 0$ large enough such that $\mathrm{dist}_{\mathcal{Y}}(T_0(t,t-T)B, A_0(t)) \leqslant \delta/2$. From (2.13) we can choose $\eta_0 > 0$ small enough such that $\|T_\eta(t,t-T)z - T_0(t,t-T)z\|_{\mathcal{Y}} \leqslant \delta/2$ for all $z \in B$ and $0 < \eta \leqslant \eta_0$. Hence, we have $\mathrm{dist}_{\mathcal{Y}}(T_\eta(t,t-T)B, A_0(t)) \leq \delta$ and from the invariance properties of the attractor $\mathcal{A}_\eta$ we have $A_\eta(t) \subset T(t,t-T)B$ and therefore $\mathrm{dist}_{\mathcal{Y}}(A_\eta(t), A_0(t)) \leqslant \delta$ for $0 < \eta \leq \eta_0$. $\square$

Without any other assumption on the structure of the attractors and in particular of the limitting attractor $\mathcal{A}_0$ it seems very unlikely to obtain stronger convergence statements than the ones from Corollary 8.2 and Corollary 8.5. In particular it does not seem possible to obtain a continuity result of the attractor, unless, roughly speaking, the atractor $\mathcal{A}_0$ is obtained, for each $t \in \mathbb{R}$, as the union of the unstable manifolds of all the hyperbolic, bounded and global solutions of (2.3). Note that this is precisely the case for autonomous gradient systems, see [24, 18, 16, 17]. If this is the case, then Corollary 8.2 will provide the lower semicontinuity result and Corollary 8.5 the upper semicontinuity one.

As a matter of fact, we can show,

Theorem 8.6. *Consider the family $\{T_\eta(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, of nonlinear evolution processes and assume Condition 2.11. Condition 2.13 and Condition 2.15 hold. Suppose that, for each $\eta \in [0,1]$ the nonlinear process $\{T_\eta(t,\tau) : t \geq \tau\}$ has a global non-autonomous pullback attractor $\mathcal{A}_\eta = \{A_\eta(t)\}_{t\in\mathbb{R}}$, satisfying* (8.5).

Assume also that (2.3) has a family of hyperbolic bounded and global solutions $\xi^*_{i,0}$, $i \in I_0$ (with I_0 being a possibly countable index set) and that the pullback attractor of (2.3), $\mathcal{A}_0 = \{A_0(t) : t \in \mathbb{R}\}$, is the closure of the union of the unstable manifolds *$W^u(\xi^*_{i,0})$ of the hyperbolic bounded global solutions $\xi^*_{i,0}$, $i \in I_0$, i.e.*

$$A_0(t) = \overline{\bigcup_{i\in I_0} W^u(\xi^*_{i,0})(t)}, \quad t \in \mathbb{R}. \tag{8.7}$$

Then the family $\mathcal{A}_\eta = \{A_\eta(t),\ t \in \mathbb{R}\}$, $0 \leq \eta \leqslant \eta_0$, is upper and lower semicontinuous at $\eta = 0$, namely

$$\mathrm{dist}_{\mathcal{Y}}(A_\eta(t), A_0(t)) + \mathrm{dist}_{\mathcal{Y}}(A_0(t), A_\eta(t)) \overset{\eta\to 0}{\longrightarrow} 0, \ \textit{for all } t \in \mathbb{R}. \tag{8.8}$$

Proof. The upper semicontinuity part follows from Proposition 8.5. The lower semicontinuity part follows from (8.3) together with the hypothesis (8.7) and the fact that $\bigcup_{i\in I_\eta} W^u(\xi^*_{i,\eta})(t) \subset A_\eta(t)$ (see Proposition 2.11 in [32]), where we denote by $\{\xi^*_{i,\eta}\}_{i\in I_\eta}$ the set of global and bounded hyperbolic solutions of (2.4). □

Now, as a direct consequence of Theorem 2.9 in [17] we get uniform convergence in bounded intervals of $\mathbb{R}$

Corollary 8.7. *Under the assumptions of Theorem 8.6,*

$$\mathrm{dist}_{\mathcal{Y}}(A_\eta(t), A_0(t)) + \mathrm{dist}_{\mathcal{Y}}(A_0(t), A_\eta(t)) \xrightarrow{\eta\to 0} 0, \tag{8.9}$$

uniformly in $I \subset \mathbb{R}$, with I any bounded interval of $\mathbb{R}$.

Under stronger assumptions on the attractor $\mathcal{A}_0 = \{A_0(t)\}_{t\in\mathbb{R}}$ we can show that the convergence in (8.8) from Theorem 8.6 is uniform in $t \in \mathbb{R}$.

Corollary 8.8. *Assume we are in the hypotheses of Theorem 8.6. Consider also the following conditions.*
i) Assume the limitting attractor $\mathcal{A}_0$ has a finite number of hyperbolic bounded and global solution $\{\xi^*_{i,0}\}_{i=1}^n$ with local unstable manifolds $W^u_{loc}(\xi^*_{i,0}, \rho)$ and that for each $\epsilon > 0$, there is a $T > 0$ such that

$$\mathrm{dist}_{\mathcal{Y}}\left(A_0(t+T), \bigcup_{i=1}^n T_0(t+T,t)(W^u_{loc}(\xi^*_{i,0},\rho)(t))\right) \leqslant \epsilon, \quad \text{uniformly in } t\in\mathbb{R} \tag{8.10}$$

ii) Assume the attraction of $\mathcal{A}_0$ depends only on the time ellapsed uniformly in $t \in \mathbb{R}$, that is, for each $R > 0$, $\delta > 0$, there exists $T = T(B, \delta)$ such that

$$\mathrm{dist}_Y\left(T_0(t, t-T)(B_R), A_0(t)\right) \leqslant \epsilon, \quad \text{uniformly in } t\in\mathbb{R} \tag{8.11}$$

where B_R is the ball in $\mathcal{Y}$ centered at the origin of radious R.

Then, the convergence in (8.8) is uniform in $t \in \mathbb{R}$. As a matter of fact, from (8.10) we obtain the lower semicontinuity uniform in $t \in \mathbb{R}$ and from (8.11) the upper semicontinuity uniform in $t \in \mathbb{R}$.

Proof. Observe that from (8.10), (8.1) and since $T_\eta(t+\tau, t)(W^u_{loc}(\xi^*_\eta, \rho)(t)) \subset A_\eta(t+T)$, we obtain

$$\limsup_{\eta\to 0}\,[\mathrm{dist}_{\mathcal{Y}}(A_0(t+T), A_\eta(t+T))] \le \epsilon, \text{ uniformly in } t\in\mathbb{R}$$

which implies that the lower semicontinuity is uniform in $t \in \mathbb{R}$.

For the upper semicontinuity we repeat the proof of Proposition 8.5 and use (8.11) to obtain the upper semicontinuity uniform in $t \in \mathbb{R}$. □

9. Some generalizations

Note that many of the results in previous sections can be generalized in several ways.

First, the parameter in the equation has been considered to be a real number $\eta \in [0, 1]$ which is, by no means, a necessary restriction. In fact we coud consider problems with generic parameters in a topological space $\eta \in \Lambda$ and replace the condition $\eta \to 0$ by $\eta \to \eta_0 \in \Lambda$. No proof above needs major changes besides notations.

Second, observe that the reference problems (2.3) and (2.4) need not to be nonlinear perturbation of autonomous problems and we could consider time–dependent operators $\mathfrak{B}_0(t)$ and $\mathfrak{B}_\eta(t)$ as well. In this situation we would consider linear evolution operators

$$\begin{aligned} &\|U_\eta^0(t,s)\|_{\mathcal{L}(\mathcal{Y})} \leqslant M(t-s)^{-\gamma}e^{-\beta(t-s)t}\\ &\|U_\eta^0(t,s)\|_{\mathcal{L}(\mathcal{Z})} \leqslant M(t-s)^{-\gamma}e^{-\beta(t-s)}\\ &\|U_\eta^0(t,s)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant M(t-s)^{-\gamma}e^{-\beta(t-s)} \end{aligned} \tag{9.1}$$

for $t > s$, instead of linear semigroups in Condition 2.11. Also, Condition 2.15 will now read

$$\begin{aligned} &\sup_{0<(t-s)\leqslant T} (t-s)^{\gamma}\|U_\eta^0(t,s) - U_0^0(t,s)\|_{\mathcal{L}(\mathcal{Z})} \leqslant \rho(\eta,T),\\ &\sup_{0<(t-s)\leqslant T} (t-s)^{\gamma}\|U_\eta^0(t,s) - U_0^0(t,s)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant \rho(\eta,T),\\ &\sup_{0<(t-s)\leqslant T} (t-s)^{\gamma}\|U_\eta^0(t,s) - U_0^0(t,s)\|_{\mathcal{L}(\mathcal{Y})} \leqslant \rho(\eta,T). \end{aligned} \tag{9.2}$$

with $\rho(\eta,T) \to 0$ as $\eta \to 0$. With these notations most variations of constants fomulae used before need obvious modifications. For example, (2.7) now is written as

$$T_\eta(t,\tau)y_0 = U_\eta^0(t,\tau)y_0 + \int_\tau^t U_\eta^0(t,s)f(s,T_\eta(s,\tau)y_0)\,ds, \quad \eta\in[0,1], \tag{9.3}$$

while (2.10) and (2.11) will now read

$$U_\eta(t,\tau)y_0 = U_\eta^0(t,\tau)y_0 + \int_\tau^t U_\eta^0(t,s)(D_yf(s,\xi_\eta^*(s)))U_\eta(s,\tau)y_0ds$$

$$V_\eta(t,\tau)y_0 = U_\eta^0(t,\tau)y_0 + \int_\tau^t U_\eta^0(t,s)(D_yf(s,\xi_0^*(s)))V_\eta(s,\tau)y_0ds.$$

In fact note that in (6.5) and in (7.3), even if we start out of linear autonomous operators, we are faced to deal with linear nonautonomous ones.

These last referred expressions, (6.5) and (7.3), show that even if the original nonlinearity does not depend on parameters, one is faced to deal with problems where the nonlinear term varies with η. Hence we could consider from the beginning nonlinear terms such that

$$\sup_{t\in\mathbb{R}}\sup_{\|y\|_{\mathcal{Y}}<r}\{\|f_\varepsilon(t,y)-f_0(t,y)\|_{\mathcal{Z}} + \|D_yf_\varepsilon(t,y)-D_yf_0(t,y)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})}\} \xrightarrow{\varepsilon\to 0} 0. \tag{9.4}$$

All proofs would remain unchanged.

Note that to cope with problems of the form

$$\dot{y} = \mathfrak{B}_\eta(t)y + f_\varepsilon(t,y), \quad y(\tau) = y_0,$$

we can then consider the parameter $\tilde{\eta} = (\eta,\varepsilon)$ and read the problem as

$$\dot{y} = \mathfrak{B}_{\tilde{\eta}}(t)y + f_{\tilde{\eta}}(t,y), \quad y(\tau) = y_0.$$

10. **General examples**

10.1. **Convergence of Resolvents and Convergence of Semigroups.** In this section we present a general framework which (with variations depending on the type of problem under consideration) enables us to obtain Conditions 2.11 and 2.15. Later we use this reasoning in Section 11.1 and in Section 11.2 but it applies to many other situations with suitable changes.

Consider a family of closed densely defined operators (possibly unbounded) with common domain $\mathfrak{B}_\eta : D(\mathfrak{B}_\eta) \subset \mathcal{Z} \to \mathcal{Z}$. Assume also that the Graph Norms in $D(\mathfrak{B}_\eta)$ give equivalent norms for all values of η and with uniform equivalence constants. Assume that

$$\sup_{\eta\in[0,1]} \|\mathfrak{B}_0\mathfrak{B}_\eta^{-1}\|_{\mathcal{L}(\mathcal{Z})} < \infty, \quad \lim_{\eta\to 0} \|\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1}\|_{\mathcal{L}(\mathcal{Z})} = 0, \tag{10.1}$$

and that there is a constant $M > 0$ and $\phi \in (0, \frac{\pi}{2})$ such that, for some $\alpha \in (0, 1]$,

$$\|(\lambda - \mathfrak{B}_\eta)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leqslant \frac{M}{1 + |\lambda|^\alpha}, \quad \|\mathfrak{B}_\eta(\lambda - \mathfrak{B}_\eta)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leqslant M(1 + |\lambda|^{1-\alpha}) \quad , \forall\lambda \in \Sigma_\phi \tag{10.2}$$

where $\Sigma_\phi = \{\lambda \in \mathbb{C} : \phi \leqslant \arg\lambda \leqslant \pi\} \cup \{0\}$. Let Γ be the boundary of Σ_ϕ oriented in the direction of decreasing imaginary part.

Remark 10.1. *Observe that the case in which the domain varies with the parameter and some fractional power space independent of η could also have been considered, although we have chosen the above framework by clarity in the exposition.*

Then, we have the following preliminary Lemma.

Lemma 10.2. *If* (10.2) *holds then, there are constants $\omega > 0$ and $M \geqslant 1$, independent of $\eta \in [0, 1]$, such that*

$$\|e^{\mathfrak{B}_\eta t}\|_{\mathcal{L}(\mathcal{Z})} \leqslant Mt^{-1+\alpha}e^{-\omega t}, \quad \|e^{\mathfrak{B}_\eta t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^1)} \leqslant Mt^{-2+\alpha}e^{-\omega t}, \ t \geqslant 0.$$

Lemma 10.3. *If $\lambda \in \rho(\mathfrak{B}_\eta) \cap \rho(\mathfrak{B}_0)$ then, the following identity holds*

$$(\lambda + \mathfrak{B}_\eta)^{-1} - (\lambda + \mathfrak{B}_0)^{-1} = \mathfrak{B}_\eta(\lambda + \mathfrak{B}_\eta)^{-1}(\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1})[I - \lambda(\lambda + \mathfrak{B}_0)^{-1}]. \tag{10.3}$$

As an immediate consequence we have that

$$\|(\lambda + \mathfrak{B}_\eta)^{-1} - (\lambda + \mathfrak{B}_0)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leqslant (M + 1)^2(1 + |\lambda|^{1-\alpha})^2\|\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1}\|_{\mathcal{L}(\mathcal{Z})}. \tag{10.4}$$

Proof: First note that $\mathfrak{B}_\eta^{-1} = (I + \lambda\mathfrak{B}_\eta^{-1})(\lambda + \mathfrak{B}_\eta)^{-1}$, $0 \leqslant \eta \leqslant 1$. Then

$$\begin{aligned}
\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1} &= (I + \lambda\mathfrak{B}_\eta^{-1})(\lambda + \mathfrak{B}_\eta)^{-1} - (I + \lambda\mathfrak{B}_0^{-1})(\lambda + \mathfrak{B}_0)^{-1} \\
&= (\lambda + \mathfrak{B}_\eta)^{-1} - (\lambda + \mathfrak{B}_0)^{-1} + \lambda\mathfrak{B}_\eta^{-1}((\lambda + \mathfrak{B}_\eta)^{-1} - (\lambda + \mathfrak{B}_0)^{-1}) \\
&\quad +\lambda\mathfrak{B}_\eta^{-1}(\lambda + \mathfrak{B}_0)^{-1} - \lambda\mathfrak{B}_0^{-1}(\lambda + \mathfrak{B}_0)^{-1} \\
&= (I + \lambda\mathfrak{B}_\eta^{-1})((\lambda + \mathfrak{B}_\eta)^{-1} - (\lambda + \mathfrak{B}_0)^{-1}) + (\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1})\lambda(\lambda + \mathfrak{B}_0)^{-1}.
\end{aligned}$$

The result now follows noting that $(I + \lambda\mathfrak{B}_\eta^{-1})^{-1} = \mathfrak{B}_\eta(\lambda + \mathfrak{B}_\eta)^{-1}$.□

With this we can prove then that (2.12) is satisfied.

Theorem 10.4. *Assume that* (10.1) *and* (10.2) *are satisfied. Then, for any* $\gamma \in [0,1]$

$$t^{\gamma+(1-\alpha)(1+\gamma)}\|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Z})} \leqslant C\|\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1}\|_{\mathcal{L}(\mathcal{Z})}^{\gamma} \quad \textit{for any } \gamma \in [0,1] \tag{10.5}$$

and

$$t^{\gamma(1-\nu)+\nu+(1-\alpha)(1+\gamma(1-\nu))}\|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^\nu)} \leqslant C\|\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1}\|_{\mathcal{L}(\mathcal{Z})}^{\gamma(1-\nu)} \quad \textit{for any } \gamma,\nu \in [0,1]. \tag{10.6}$$

Proof: In fact, from (10.4), we obtain that

$$\begin{aligned}\|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Z})} &\leqslant \frac{1}{2\pi}\int_\Gamma |e^{\lambda t}|\|(\lambda - \mathfrak{B}_\eta)^{-1} - (\lambda - \mathfrak{B}_0)^{-1}\|_{\mathcal{L}(\mathcal{Z})} d|\lambda| \\ &\leqslant C\int_\Gamma e^{-\frac{1}{\cos\phi}|\mu|}(1+|\mu|^{1-\alpha})^2 d|\mu| t^{-1-2(1-\alpha)}\|\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1}\|_{\mathcal{L}(\mathcal{Z})} \\ &\leqslant Ct^{-1-2(1-\alpha)}\|\mathfrak{B}_\eta^{-1} - \mathfrak{B}_0^{-1}\|_{\mathcal{L}(\mathcal{Z})}\end{aligned}$$

and from (10.2) we have that

$$\begin{aligned}\|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Z})} &\leqslant \frac{1}{2\pi}\int_\Gamma |e^{\lambda t}|\|(\lambda - \mathfrak{B}_\eta)^{-1} - (\lambda - \mathfrak{B}_0)^{-1}\|_{\mathcal{L}(\mathcal{Z})} d|\lambda| \\ &\leqslant C\int_\Gamma e^{-\frac{1}{\cos\phi}|\mu|}\frac{2M}{1+|\mu|^\alpha} d|\mu| t^{-1+\alpha} \\ &\leqslant Ct^{-1+\alpha}.\end{aligned}$$

From this (10.5) follows easily. As in the previous estimate

$$\begin{aligned}\|e^{\mathfrak{B}_\eta t} - e^{\mathfrak{B}_0 t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^1)} &\leqslant \frac{1}{\pi}\int_\Gamma |e^{\lambda t}|[\|\mathfrak{B}_0\mathfrak{B}_\eta^{-1}\mathfrak{B}_\eta(\lambda - \mathfrak{B}_\eta)^{-1}\|_{\mathcal{L}(\mathcal{Z})} + \|\mathfrak{B}_0(\lambda - \mathfrak{B}_0)^{-1}\|_{\mathcal{L}(\mathcal{Z})}] d|\lambda| \\ &\leqslant C\int_\Gamma e^{-\frac{1}{\cos\phi}|\mu|}(1+|\mu|^{1-\alpha}) d|\mu| t^{-2+\alpha} \\ &\leqslant Ct^{-2+\alpha}.\end{aligned}$$

From this, (10.5) and the Moment's inequality (10.6) follows.□

10.2. **Asymptotically autonomous gradient systems.** In [15] (see also [18]), a similar situation than in this paper was considered. In that case $\mathcal{Y} = \mathcal{Z}$ and the semigroups $e^{\mathfrak{B}_\eta t}$ are assumed to be strongly continuous in $\mathcal{Z}$. Then the authors prove that the pullback attractor resulting from a suitably small non-autonomous perturbation of an autonomous gradient like semigroup in which there are a finite number of equilibria, all of them hyperbolic, has a similar structure, namely for $\eta \to 0$,

$$A_\eta(t) = \bigcup_{j=1}^n W^u(\xi_{j,\eta}(\cdot))(t), \quad t \in \mathbb{R}, \tag{10.7}$$

where the $\xi_{j,\eta}(\cdot)$ are global hyperbolic bounded solutions (see [39] and [23] for similar results, the first one related to regular non-autonomous perturbations of semigroups, and the second one for a singular perturbation in the nonlinear part of equations. Note that our singular perturbation of unbounded operators differs crucially from all the above references). The proof is based on the convergence of processes to the corresponding gradient semigroup as in (2.13). Moreover, it is proved in [15] that actually $A_\eta(t)$ is a uniform pullback (and so forwards) exponential attractor.

Finally, putting together the main results from this work and from [14, 15] we conclude the continuity of attractors under both singular and regular perturbation, i.e., consider the autonomous semilinear problem on the Banach space $\mathcal{Y}$.

$$\dot{y} = \mathfrak{B}_0 y + f_0(y), \quad y(\tau) = y_0 \tag{10.8}$$

and, for $\varepsilon \in [0,1]$, a singular (in the hypotheses of Section 1) and non-autonomous perturbation of it

$$\dot{y} = \mathfrak{B}_\eta y + f_\varepsilon(t,y), \quad y(\tau) = y_0, \tag{10.9}$$

where, for each $r > 0$,

$$\sup_{t\in\mathbb{R}} \sup_{\|y\|_{\mathcal{Y}}<r} \{\|f_\varepsilon(t,y) - f_0(y)\|_{\mathcal{Z}} + \|D_y f_\varepsilon(t,y) - D_y f_0(y)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})}\} \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{10.10}$$

Then, if (10.8) is a gradient system, we prove the continuity and characterization of the pullback attractors associated to (10.9).

Let $\{T_{\eta,\varepsilon}(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ be the nonlinear evolution process in $\mathcal{Y}$ associated to (10.9). Assume that $\{T_{\eta,\varepsilon}(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ has a pullback attractor $\{A_{\eta,\varepsilon}(t) : t \in \mathbb{R}\}$. Assume also that the nonlinear semigroup $\{T_0 = T_{0,0}(t) : t \geqslant 0\}$, associated to (10.8), is Lipschitz continuous on bounded sets; that is, for each bounded subset B of $\mathcal{Y}$, there exist $c = c(B)$ and $L = L(B) > 0$ such that

$$\|T_{\eta,\epsilon}(t)u - T_{\eta,\epsilon}(t)v\|_{\mathcal{Y}} \leqslant ce^{Lt}\|u - v\|_{\mathcal{Y}}, \text{ for all } u, v \in B, \ \forall \eta \in [0,1], \ \forall \epsilon \in [0,1]. \tag{10.11}$$

Then,

Theorem 10.5. *Let $\{A_{\eta,\varepsilon}(t) : t \in \mathbb{R}\}$ be the pullback attractor for $\{T_{\eta,\varepsilon}(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$. Assume that $\{T_0(t) : t \geqslant 0 \in \mathbb{R}\}$ is a gradient system (that is, it has a Lyapunov function), has a finite number of isolated hyperbolic equilibria $\mathcal{S} = \{y_1^*, \cdots, y_n^*\}$ and that (10.10) is satisfied. Then the family $\{A_{\eta,\varepsilon}(t), 0 \leq \eta \leqslant \eta_0\}$ is upper and lower semicontinuous at $\eta, \varepsilon = 0$, namely*

$$\sup_{t\in\mathbb{R}} \mathrm{dist_H}(A_{\eta,\varepsilon}(t), A_0(t)) \overset{\eta,\varepsilon\to 0}{\longrightarrow} 0, \tag{10.12}$$

where $A_0(t) = A$ for all $t \in \mathbb{R}$ is the global attractor for $\{T_0(t) : t \geqslant 0\}$. Moreover, there exists $\eta_0, \varepsilon_0 > 0$ such that

$$A_{\eta,\varepsilon}(t) = \cup_{i=1}^n W^u(\xi_{i,\eta,\varepsilon}^*)(t), \ \forall t \in \mathbb{R} \text{ and } \forall \eta \in [0,\eta_0] \text{ and } \varepsilon \in [0,\varepsilon_0],$$

being $\xi_{i,\eta,\varepsilon}^*$ global hyperbolic solutions of (10.9).

Finally, there exists $\gamma > 0$ such that, for $B \subset \mathcal{Z}$ bounded,

$$\sup_{\tau\in\mathbb{R}} \mathrm{dist}(T_{\eta,\varepsilon}(t+\tau,\tau)u_0, A_{\eta,\varepsilon}(t+\tau)) \leqslant c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \tag{10.13}$$

Proof. The proof of the continuity of the η, ε-attractors is just a consequence of Theorem 8.6 and Theorem 7.1 in [14]. Indeed, consider

$$\dot{y} = \mathfrak{B}_0 y + f_0(y), \quad y(\tau) = y_0 \tag{10.14}$$

and an autonomous perturbation of it

$$\dot{y} = \mathfrak{B}_\eta y + f_0(y), \quad y(\tau) = y_0, \tag{10.15}$$

and next, consider a non-autonomous perturbation of (10.15)

$$\dot{y} = \mathfrak{B}_\eta y + f_\varepsilon(t, y), \quad y(\tau) = y_0. \tag{10.16}$$

Now, as a consequence of our results and the results in [15], the semigroup associated to (10.15) is gradient like. Now, from [15], a non-autonomous perturbation of a gradient semigroup is a gradient-like process, and the pullback attractor of the process associated to (10.16) is such that

$$\mathcal{A}_{\eta,\varepsilon}(t) = \cup_{i=1}^n W^u(\xi^*_{i,\eta,\varepsilon})(t), \ \forall t \in \mathbb{R} \text{ and } \forall \eta \in [0, \eta_0].$$

holds for all ε suitably small.

Finally, the exponential attraction property in (10.13) follows as in the case of regular perturbations (see [15]). □

11. Application to concrete problems

This section is devoted to present a few examples of diverse nature for which the theory developed here applies. Each example has a very peculiar singular nature and where chosen to show the variety of singular perturbation problems that appear in the modeling of real world phenomena by differential equations

11.1. Varying Diffusivity.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 1$ and $\epsilon \in [0, 1]$. Consider the parabolic problem

$$\begin{aligned} & u_t = \operatorname{div}(a_\epsilon \nabla u) - u + f(t, u), \quad x \in \Omega \\ & \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \end{aligned} \tag{11.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function with bounded derivatives up to second order and $a_\epsilon : \bar{\Omega} \to [1, 2]$ is a continuously differentiable function.

Assume that $a_\epsilon \overset{\epsilon \to 0}{\longrightarrow} a_0$ in $L^1(\Omega)$ and let $A_\epsilon : D(A_\epsilon) \subset L^2(\Omega) \to L^2(\Omega)$ be the linear operator defined by

$$\begin{aligned} D(A_\epsilon) = \{u \in H^2(\Omega) : \ \frac{\partial u}{\partial n} = 0\} := H^2_n(\Omega), \\ A_\epsilon u = -\operatorname{div}(a_\epsilon \nabla u) + u, \quad \forall u \in D(A_\epsilon). \end{aligned}$$

Note that,

Theorem 11.1. *There is a constant $M > 0$, independent of ϵ, such that*

$$\|(\lambda + A_\epsilon)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \leqslant \frac{M}{1 + |\lambda|}$$

for all $\lambda \in \Sigma_\phi$ and for each $\phi \in (0, \frac{\pi}{2})$.

Proof: That follows immediately from the fact that A_ϵ is positive, self adjoint and its first eigenvalue λ_1^ϵ converges to the first eigenvalue λ_1^0 of A_0.□

From this and Lemma 10.2 with $\alpha = 1$ we immediately have that

Lemma 11.2. *If* (10.2) *holds then, there are constants $\omega > 0$ and $M \geqslant 1$, independent of $\epsilon \in [0,1]$, such that with $\mathcal{Z} = L^2(\Omega)$ we have*

$$\|e^{A_\epsilon t}\|_{\mathcal{L}(\mathcal{Z})} \leqslant Me^{-\omega t}, \quad \|e^{A_\epsilon t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^1)} \leqslant Mt^{-1}e^{-\omega t},\ t > 0,$$
$$\|e^{A_\epsilon t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^\nu)} \leq Mt^{-\nu}e^{-\omega t}, \quad \|e^{A_\epsilon t}\|_{\mathcal{L}(\mathcal{Z}^\nu)} \leqslant Mt^{-\nu}e^{-\omega t},\ t > 0.$$

This verifies Condition 2.11.

To verify Condition 2.13 we use the results of Section 10.1. Let us verify (10.1). Clearly, from our hypothesis,

$$\sup_{\epsilon\in[0,1]} \|A_0 A_\epsilon^{-1}\|_{\mathcal{L}(L^2(\Omega))} < \infty.$$

In order to prove that A_ϵ^{-1} converges in the uniform operator topology to A_0^{-1} we note that $u_\epsilon = A_\epsilon^{-1} f_\epsilon$ means that u_ϵ satisfies

$$\int_\Omega a_\epsilon \nabla u_\epsilon \nabla \phi + \int_\Omega u_\epsilon \phi = \int_\Omega f_\epsilon \phi, \text{ for each } \phi \in H^1(\Omega) \tag{11.2}$$

It is easy to see that, if $\|f_\epsilon\|_{L^2(\Omega)} \leqslant 1$ then there is a constant $c > 0$ such that $\|u_\epsilon\|_{H^1(\Omega)} \leqslant c$. From this we have that there is a subsequence (which we denote the same) such that $u_\epsilon \overset{\epsilon\to 0}{\longrightarrow} u_0$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$.

Also, since $a_\epsilon \to a_0$ in $L^1(\Omega)$ there is a subsequence which converges almost everywhere in Ω. From the Lebesgue Dominated Convergence Theorem, $a_\epsilon \nabla\phi$ converges to $a_0\nabla\phi$ in $L^2(\Omega)$. Since the limit is independent of the subsequence taken the convergence follows. Making $\epsilon \to 0$ in (11.2) we have that, if $f_\epsilon \to f_0$ weakly in $L^2(\Omega)$.

With this we prove that

$$\int_\Omega a_0 \nabla u_0 \nabla \phi + \int_\Omega u_0 \phi = \int_\Omega f_0 \phi, \text{ for each } \phi \in H^1(\Omega) \tag{11.3}$$

and, consequently, $u_0 = A_0^{-1} f_0$.

Remark 11.3. *Asking that $a_\epsilon \overset{\epsilon\to 0}{\longrightarrow} a_0$ uniformly in $\bar{\Omega}$ and that $f_\epsilon \overset{\epsilon\to 0}{\longrightarrow} f_0$ weakly in $L^2(\Omega)$, we can also prove the continuity of the solutions u_ϵ of* (11.2) *to the solution u_0 of* (11.3) *strongly in $H^1(\Omega)$. That is accomplished using the variational formulation of the solution.*

In fact we can prove

Theorem 11.4. $\|A_\epsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(\mathcal{Z})} \overset{\epsilon\to 0}{\longrightarrow} 0.$

Proof: Suppose not. Then, there exists a sequence $u_n \in L^2(\Omega)$, $\|u_n\|_{L^2(\Omega)} = 1$ and $d > 0$ such that

$$\|A_\epsilon^{-1} u_n - A_0^{-1} u_n\|_{L^2(\Omega)} \geqslant d.$$

Hence there is a subsequence, which we again denote by u_n, and a function u such that $u_n \overset{n\to\infty}{\rightharpoonup} u$. From our previous result $A_\epsilon^{-1}u_n \to A_0^{-1}u$ which is a contradiction. □

From this (10.1) is satisfied. It follows from (10.4) with $\alpha = 1$ that

Theorem 11.5. *Assume that* (10.1) *and* (10.2) *are satisfied. Then, for any* $\gamma \in [0,1]$

$$t^\gamma \|e^{A_\epsilon t} - e^{A_0 t}\|_{\mathcal{L}(\mathcal{Z})} \leqslant C\|A_\epsilon^{-1} - A_0^{-1}\|^\gamma_{\mathcal{L}(\mathcal{Z})} \tag{11.4}$$

and, for any $\gamma, \nu \in [0,1]$,

$$t^{\gamma(1-\nu)+\nu} \|e^{A_\epsilon t} - e^{A_0 t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^\nu)} \leqslant C\|A_\epsilon^{-1} - A_0^{-1}\|^{\gamma(1-\nu)}_{\mathcal{L}(\mathcal{Z})}. \tag{11.5}$$

and

$$t^{\gamma(1-\nu)+\nu} \|e^{A_\epsilon t} - e^{A_0 t}\|_{\mathcal{L}(\mathcal{Z}^\nu)} \leqslant C\|A_\epsilon^{-1} - A_0^{-1}\|^{\gamma(1-\nu)}_{\mathcal{L}(\mathcal{Z})}. \tag{11.6}$$

This completes the verification of Condition 2.15.

Now, if in Theorem 11.5 we take $\gamma < \frac{1}{2}$ and $\nu = \frac{1}{2}$, the Nemitskiui map associated to $f(t,\cdot) : \mathcal{Z}^{\frac{1}{2}} \to \mathcal{Z}$ clearly satisfies Condition 2.13.

11.2. **A semilinear problem with singularity at the inital time.** Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and P_0, P_1 be arbitrary points in $\bar{\Omega}$. Consider the problem

$$\begin{cases} w_t - \operatorname{div}(a_\epsilon \nabla w) + w = f_1(t,w) & x \in \Omega,\ t > 0 \\ \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ v_t - \dfrac{1}{g}\,(g v_x)_x + v = f_2(t,v), & x \in (0,1) \\ v(0) = w(P_0), \quad v(1) = w(P_1) \end{cases} \tag{11.7}$$

where $a_\epsilon : \Omega \to [1,2]$, $\epsilon \geqslant 0$, is $C^1(\bar{\Omega})$, $a_\epsilon \to a_0$ in $L^1(\Omega)$ and $f_1(\cdot,\cdot), f_2(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are bounded C^1 functions which are globally Lipschitz continuous in the second variable uniformly for $t \in \mathbb{R}$.

Following [5], consider the Banach space $U_0^p := L^p(\Omega) \oplus L^p_g(0,1)$, that is $(w,v) \in U_0^p$ if $w \in L^p(\Omega)$, $v \in L^p(0,1)$ and the norm is given by

$$\|(w,v)\|^p_{U_0^p} = \int_\Omega |w|^p + \int_0^1 g|v|^p.$$

For $\frac{N}{2} < q \leqslant p$, let $\mathcal{Y} = U_0^p$ and $\mathcal{Z} = U_0^q$ and let $A_\epsilon : D(A_\epsilon) \subset \mathcal{Y} \to \mathcal{Y}$ be defined by

$$D(A_\epsilon) = \big\{(w,v) \in \mathcal{Y} : w \in D(\Lambda_\epsilon),\ (gv')' \in L^p(0,1),\ v(0) = w(P_0),\ v(1) = w(P_1)\big\}$$

$$A_\epsilon(w,v) = \Big(-\operatorname{div}(a_\epsilon \nabla w) + w, -\frac{1}{g}\,(gv')' + v\Big), \quad (w,v) \in D(A_\epsilon), \tag{11.8}$$

where where $D(\Lambda_\epsilon) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\}$ and $\Lambda_\epsilon u = -\operatorname{div}(a_\epsilon \nabla w) + w$, for all $w \in D(\Lambda_\epsilon)$. Moreover, for $p > N/2$ we have from Lemma B.1 v) in [4] that $D(\Lambda_\epsilon)$ is continuously embedded in $C^r(\bar{\Omega})$ for some $r \in (0,1)$ and with uniform embedding constant. This tells us that the functions in $D(\Lambda_\epsilon)$ have trace at P_0 and P_1.

Proposition 11.6. *The operator A_ϵ defined by* (11.8) *has the following properties*
(i) $D(A_\epsilon)$ is dense in $\mathcal{Y}$,
(ii) A_ϵ is a closed operator,
(iii) A_ϵ has compact resolvent and
(iv) $\rho(A_\epsilon) \supset \Sigma_\theta$, where Σ_θ is given by

$$\Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : \ |\arg\lambda| \leqslant \pi - \theta\} \cup \{0\} \tag{11.9}$$

and we have the following estimates

$$\|(A_\epsilon + \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leqslant \frac{C}{|\lambda|^\alpha + 1} \tag{11.10}$$

$$\|(A_\epsilon + \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Y})} \leqslant \frac{C}{|\lambda|^\alpha + 1} \tag{11.11}$$

$$\|(A_\epsilon + \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant \frac{C}{|\lambda|^\alpha + 1} \tag{11.12}$$

and

$$\|A_\epsilon(A_\epsilon + \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Y})} \leqslant C(1 + |\lambda|^{1-\alpha}). \tag{11.13}$$

for each $0 < \alpha < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{p} - \frac{1}{q}) < 1$, $\lambda \in \Sigma_\theta$.

Remark 11.7. *We note that α can be taken as close to 1 as we wish by choosing p and q large.*

Consequently, as in Lemma 10.2, we have that

Lemma 11.8. *From* (11.10), (11.11), (11.12) *and* (11.13), *there are constants $\omega > 0$ and $M \geqslant 1$, independent of $\epsilon \in [0, 1]$, such that*

$$\|e^{A_\epsilon t}\|_{\mathcal{L}(\mathcal{Z})} \leqslant Mt^{\alpha-1}e^{-\omega t}, \quad \|e^{A_\epsilon t}\|_{\mathcal{L}(\mathcal{Y})} \leqslant Mt^{\alpha-1}e^{-\omega t}, \quad \|e^{A_\epsilon t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant Mt^{\alpha-1}e^{-\omega t}, \ t > 0.$$

This completes the verification of Condition 2.11.

Proceeding as in Lemma 10.3 and using Proposition 11.6 we have that

Lemma 11.9. *If $\lambda \in \rho(A_\epsilon) \cap \rho(A_0)$ then, the following identity holds*

$$(\lambda + A_\epsilon)^{-1} - (\lambda + A_0)^{-1} = A_\epsilon(\lambda + A_\epsilon)^{-1}(A_\epsilon^{-1} - A_0^{-1})[I - \lambda(\lambda + A_0)^{-1}]. \tag{11.14}$$

As an immediate consequence we have that

$$\begin{aligned}
&\|(\lambda + A_\epsilon)^{-1} - (\lambda + A_0)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leqslant C(1 + |\lambda|^{1-\alpha})^2\|A_\epsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(\mathcal{Z})},\\
&\|(\lambda + A_\epsilon)^{-1} - (\lambda + A_0)^{-1}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant C(1 + |\lambda|^{1-\alpha})^2\|A_\epsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(\mathcal{Z})},\\
&\|(\lambda + A_\epsilon)^{-1} - (\lambda + A_0)^{-1}\|_{\mathcal{L}(\mathcal{Y})} \leqslant C(1 + |\lambda|^{1-\alpha})^2\|A_\epsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(\mathcal{Z})}.
\end{aligned} \tag{11.15}$$

Also,

Theorem 11.10. $\|A_\epsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(\mathcal{Z})} \overset{\epsilon\to 0}{\longrightarrow} 0.$

Proof: Let $(f,h)\in\mathcal{Y}$. Solving the equation $(w_\epsilon,v_\epsilon)=A_\epsilon^{-1}(f,h)$ is equivalent to solve $A_0(w_\epsilon,v_\epsilon)=(f,h)$, which is equivalent to find the functions (w_ϵ,v_ϵ) verifying,

$$\begin{cases} -\mathrm{div}(a_\epsilon\nabla w_\epsilon)+w_\epsilon=f, & x\in\Omega\\ \dfrac{\partial w_\epsilon}{\partial n}=0, & x\in\partial\Omega\\ \\ -\dfrac{1}{g}\,(g\,v_\epsilon')'+v_\epsilon=h, & s\in(0,1),\\ \quad v_\epsilon(0)=w_\epsilon(P_0),\quad v_\epsilon(1)=w_\epsilon(P_1). \end{cases} \tag{11.16}$$

Clearly, from Theorem 11.4 and from (11.16) we have that $\|w_\epsilon-w\|_{L^2(\Omega)}\stackrel{\epsilon\to0}{\longrightarrow}0$ and consequently (using again Lemma B.1 v) of [4]) $\|w_\epsilon-w\|_{C(\bar\Omega)}\stackrel{\epsilon\to0}{\longrightarrow}0$ for any $s<2$.

We consider now the change of variables $z=v_\epsilon-\xi_\epsilon$, where ξ_ϵ is the solution of the following problem

$$\begin{cases} -\dfrac{1}{g}\,(g\,\xi_\epsilon')'=0,\quad s\in(0,1)\\ \quad\xi_\epsilon(0)=w_\epsilon(P_0),\quad \xi_\epsilon(1)=w_\epsilon(P_1). \end{cases} \tag{11.17}$$

and we apply it to the last two equations of (11.16), we have

$$\begin{cases} -\dfrac{1}{g}\,(g\,z_\epsilon')'+z_\epsilon=h-\xi_\epsilon,\quad s\in(0,1)\\ \quad z_\epsilon(0)=z_\epsilon(1)=0. \end{cases}$$

Note that, if $A_g:D(A_g)\subset L^p_g(0,1)\to L^p_g(0,1)$ is the operator given by

$$\begin{aligned} &D(A_g)=\{z\in L^p_g(0,1):(gz')'\in L^p_g(0,1):z(0)=z(1)=0\}\\ &A_gz=-\frac{1}{g}\,(g\,z')'+z,\ \forall z\in D(A_g), \end{aligned}$$

we have the following resolvent estimates

$$\|z_\epsilon-z_0\|_{L^p_g(0,1)}=\|A_g^{-1}(\xi_\epsilon-\xi_0)\|_{L^p_g(0,1)}\leqslant C\|\xi_\epsilon-\xi_0\|_{L^q_g(0,1)} \tag{11.18}$$

Hence, for $v_\epsilon=z_\epsilon+\xi_\epsilon$ we have, for $s<2$ suitably close to 2.

$$\|v_\epsilon-v_0\|_{L^p_g(0,1)}\leqslant C\|\xi_\epsilon-\xi_0\|_{L^q_g(0,1)}\le C\|w_\epsilon-w_0\|_{C(\bar\Omega)}\leqslant C\|w_\epsilon-w_0\|_{W^{s,p}(\Omega)}.$$

This concludes the proof of the theorem. □

With this we can prove in a similar way as in Theorem 10.4 that

Theorem 11.11. *For p suitably large, there are $\gamma<\frac12$ and $r(\gamma)>0$ such that*

$$t^\gamma\|e^{A_\epsilon t}-e^{A_0t}\|_{\mathcal{L}(\mathcal{Z})}\leqslant C\|A_\epsilon^{-1}-A_0^{-1}\|^{r(\gamma)}_{\mathcal{L}(\mathcal{Z})} \tag{11.19}$$

and

$$t^\gamma\|e^{A_\epsilon t}-e^{A_0t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Z}^\alpha)}\leqslant C\|A_\epsilon^{-1}-A_0^{-1}\|^{r(\gamma)}_{\mathcal{L}(\mathcal{Z})}. \tag{11.20}$$

This completes the verification of Condition 2.15.

11.3. **Second order dissipative ode.** Following [12] we consider the Cauchy problem for the following second order ordinary differential equation

$$\begin{aligned} &\epsilon\ddot{x}+\dot{x}=-\mu x+f(t,x),\ (t,x)\in\mathbb{R}\times\mathbb{R}^n\\ &x(\tau)=x_0\in\mathbb{R}^n,\ x_t(\tau)=v_0\in\mathbb{R}^n. \end{aligned} \tag{11.21}$$

assume that $\mu>0$, $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is a C^1 function which is globally Lipschitz, globally bounded and that there is a $M>0$ and with symmetric Jacobian matrix at every point. If we rewrite the above equation in the form of a system with variables x and $v=\epsilon\dot{x}$ we have that

$$\begin{aligned} &\frac{d}{dt}\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]=\begin{bmatrix}0 & I/\epsilon\\ -\mu & -I/\epsilon\end{bmatrix}\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]+\left[\begin{smallmatrix}0\\f(t,x)\end{smallmatrix}\right],\quad \left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]\in\mathbb{R}^n\times\mathbb{R}^n\\ &\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right](\tau)=\left[\begin{smallmatrix}x_0\\\epsilon v_0\end{smallmatrix}\right]\in\mathbb{R}^n\times\mathbb{R}^n. \end{aligned} \tag{11.22}$$

Clearly, the solutions for (11.22) are globally defined. If, for each $\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]\in\mathbb{R}^n\times\mathbb{R}^n$, $S_\epsilon(t,\tau)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]$ denotes the solution of (11.22) at time t, the solution operator family $\{S_\epsilon(t,\tau):t\geqslant\tau\}$ defines an evolution process in $\mathcal{Y}=\mathcal{Z}=\mathbb{R}^n\times\mathbb{R}^n$ with the norm $\|\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]\|_{\mathcal{Y}}^2=\mu x^2+v^2$.

The above system can be rewritten as

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix}I & I\\ -\epsilon\mu I & 0\end{bmatrix}\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]=-\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]+\begin{bmatrix}I & I\\ -\epsilon\mu I & 0\end{bmatrix}\begin{bmatrix}0\\ f(t,x)\end{bmatrix},\quad \left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]\in\mathbb{R}^n\times\mathbb{R}^n\\ &\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right](\tau)=\left[\begin{smallmatrix}x_0\\\epsilon v_0\end{smallmatrix}\right]\in\mathbb{R}^n\times\mathbb{R}^n. \end{aligned} \tag{11.23}$$

As the parameter ϵ tends to zero, one would expect that the dynamical properties of (11.22) would be given by

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix}I & I\\ 0 & 0\end{bmatrix}\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]=-\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]+\begin{bmatrix}I & I\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ f(t,x)\end{bmatrix},\quad \left[\begin{smallmatrix}x\\v\end{smallmatrix}\right]\in\mathbb{R}^n\times\mathbb{R}^n\\ &\left[\begin{smallmatrix}x\\v\end{smallmatrix}\right](\tau)=\left[\begin{smallmatrix}x_0\\0\end{smallmatrix}\right]\in\mathbb{R}^n\times\mathbb{R}^n. \end{aligned} \tag{11.24}$$

which corresponds to $v=0$ and

$$\begin{aligned} &\frac{d}{dt}x=-\mu x+f(t,x)\\ &x(\tau)=x_0\in\mathbb{R}^n. \end{aligned} \tag{11.25}$$

Note that the solutions for (11.25) are globally defined and the solution operator family $\{R_0(t,\tau):t\geqslant\tau\}$ defines an evolution process in $\mathbb{R}^n$. To compare the dynamics of these two problems we should find a way to see the dynamics of (11.25) in X. That is done simply defining

$$S_0(t,\tau)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]=\left[\begin{smallmatrix}R_0(t,\tau)x_0\\0\end{smallmatrix}\right],t>\tau \text{ and } S_0(\tau,\tau)=I,\ \tau\in\mathbb{R}.$$

and noting that

1) $S_0(t,\tau)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]=\left[\begin{smallmatrix}R_0(t,\tau)x_0\\0\end{smallmatrix}\right]=\left[\begin{smallmatrix}R_0(t,s)R_0(s,\tau)x_0\\0\end{smallmatrix}\right]=S_0(t,s)\left[\begin{smallmatrix}R_0(s,\tau)x_0\\0\end{smallmatrix}\right]=S_0(t,s)S_0(s,\tau)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]$ for all $t\geqslant s\geqslant\tau$;
2) $(t,\tau,\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right])\mapsto S_0(t,\tau)\left[\begin{smallmatrix}x_0\\v_0\end{smallmatrix}\right]\in\mathcal{Y}$ is continuous.

Consequently, $\{S_0(t,\tau) : t \geqslant \tau\}$ is a singular evolution process at zero.

The linear semigroup $\{e^{A_\epsilon t} : t \geqslant 0\}$ associated to

$$\begin{aligned} &\frac{d}{dt}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = \begin{bmatrix} 0 & I/\epsilon \\ -\mu I & -I/\epsilon \end{bmatrix}\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathbb{R}^n \times \mathbb{R}^n \\ &\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} x_0 \\ \epsilon v_0 \end{smallmatrix}\right] \in \mathbb{R}^n \times \mathbb{R}^n. \end{aligned} \tag{11.26}$$

is given by

$$T_\epsilon(t) = \begin{bmatrix} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} & \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\epsilon(\lambda_1 - \lambda_2)} \\ \frac{\epsilon \lambda_1 \lambda_2 (e^{\lambda_2 t} - e^{\lambda_1 t})}{\lambda_1 - \lambda_2} & \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \end{bmatrix}$$

where $\lambda_1 = \frac{-1+\sqrt{1-4\epsilon\mu}}{2\epsilon}$ and $\lambda_2 = \frac{-1-\sqrt{1-4\epsilon\mu}}{2\epsilon}$, $\epsilon(\lambda_1 - \lambda_2) = \sqrt{1-4\epsilon\mu}$. Note that, $\lambda_1 \overset{\epsilon\to 0}{\longrightarrow} -\mu$, $\lambda_2 \overset{\epsilon\to 0}{\longrightarrow} -\infty$, $\epsilon\lambda_1 \overset{\epsilon\to 0}{\longrightarrow} 0$, $\epsilon\lambda_2 \overset{\epsilon\to 0}{\longrightarrow} -1$.

Define the singular semigroup $\{T_0(t) : t \geqslant 0\}$ by

$$T_0(t) = \begin{bmatrix} e^{-\mu t} & e^{-\mu t} \\ 0 & 0 \end{bmatrix}, \ t > 0.$$

Theorem 11.12. *Let $\mathcal{Y} = \mathcal{Z} := \mathbb{R}^n \times \mathbb{R}^n$ with the norm $\|\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\|_{\mathcal{Z}}^2 = \|v\|_{\mathbb{R}^n}^2 + \mu\|u\|_{\mathbb{R}^n}^2$. There exists a constant $M \geqslant 1$, independent of $\epsilon \in [0,1]$, and of $\mu \geqslant 1$, such that*

$$\|T_\epsilon(t)\|_{\mathcal{L}(\mathcal{Y})} \leqslant M \tag{11.27}$$

and, for $\alpha \in [0,1]$,

$$\|T_\epsilon(t) - T_0(t)\|_{\mathcal{L}(\mathcal{Y})} \leqslant c\mu^{\frac{1+\alpha}{4}} \epsilon^\alpha t^{-\frac{1+\alpha}{2}} e^{-t\frac{1-\alpha}{6}}. \tag{11.28}$$

This ensures that Condition 2.11 and Condition 2.15 are satisfied. Clearly, Condition 2.13 is also satisfied.

11.4. **Viscous Cahn-Hilliard equation.** Following [13] consider the viscous Cahn-Hilliard problem

$$\begin{aligned} &(1-\nu)u_t = -\Delta(\Delta u + f(t,u) - \nu u_t), \text{ in } \Omega, \\ &u(t,x) = \Delta u(t,x) = 0 \text{ in } \partial\Omega, \\ &u(0,x) = u_0(x), \end{aligned} \tag{11.29}$$

where $\nu \in [0,1]$, $f \in C^1(\mathbb{R},\mathbb{R})$ is a bounded function with bounded derivatives up to second order and Ω is a bounded smooth domain in $\mathbb{R}^n, n \geqslant 3$.

Let $A = -\Delta$ with Dirichlet boundary condition, $D(A) = H_0^1(\Omega) = X^1$ and A takes values in $X = H^{-1}(\Omega) = (H_0^1(\Omega))'$. Denote by $X^\alpha := D(A^\alpha), \alpha \geqslant 0$, fractional power spaces associated with A. Define $A_\nu = A^2((1-\nu)I + \nu A)^{-1}$ and let $B_\nu = A((1-\nu)I + \nu A)^{-1}$. Clearly, for all $\nu \in [0,1]$, A_ν are self-adjoint and positive operators defined in $D(A)$ with values in X for $\nu \in (0,1]$ and with values in X^{-1}) if $\nu = 0$; X^{-1} being the extrapolated space. Similarly, the operator B_ν is bounded positive and self-adjoint if $\nu \in (0,1]$ and $B_0 = A : D(A) \subset X \to X$.

Using this notation we rewrite (11.29) as

$$u_t = -A_\nu u + B_\nu f(t,u), \tag{11.30}$$

or, since $A_\nu^{-1}B_\nu = A^{-1}$, as

$$\frac{d}{dt}\left(A_\nu^{-1}u\right) = -u + A^{-1}f(t,u). \tag{11.31}$$

Now, from the convegence

$$A_\nu^{-1} = ((1-\nu)I + \nu A)A^{-2} \to A^{-1} \quad \text{as} \ \ \nu \to 1^-,$$
$$A_\nu^{-1} \to A^{-2} \quad \text{as} \ \ \nu \to 0^+$$

in the uniform operator topology, the limit problem $(11.31)_{\nu=1}$ will be the semilinear heat equation, while $(11.31)_{\nu=0}$ will be the classical Cahn-Hilliard equation.

Hereafter we assume, without loss of generality, that $f(t,0)=0$. In fact, if $f(t,0)\neq 0$ we may replace $f(t,s)$ by $\tilde f(t,s) = f(t,s) - f(t,0)$ without changing (11.29).

The equation (11.30) defines a process $\{S(t,\tau) : t \geqslant \tau\}$ in $\mathcal{Y} := X^1$, $\nu \in [0,1]$ and

$$S_\nu(t,\tau)u_0 = e^{-A_\nu(t-\tau)}u_0 + \int_\tau^t B_\nu e^{-A_\nu(t-s)} f(s, S_\nu(s)u_0)ds$$

and

$$S_0(t,\tau)u_0 = e^{-A^2(t-\tau)}u_0 + \int_\tau^t A e^{-A^2(t-s)} f(s, S_0(s)u_0)ds.$$

Let $\mathcal{Z} = X^\epsilon$, $0 < \epsilon < \frac{1}{2}$. Clearly, $f : \mathcal{Y} \mapsto \mathcal{Z}$, satisfies Condition 2.13. Also, it is easy (from the fact that A_ν is self adjoint and positive) that Condition 2.11 is satisfied. Hence we only need to verify Condition 2.15. To that end, note that

Lemma 11.13 ([13]). *There is a $\phi \in (0, \frac{\pi}{2})$ and a constant $M > 0$ such that*

$$\|A_\nu^{-1} - A^{-2}\| \leqslant M\nu, \quad \|A_\nu^{-1} - A_\mu^{-1}\| \leqslant M|\nu - \mu| \text{and}$$

$$\|(\lambda + A_\nu)^{-1}\| \leqslant \frac{M}{1+|\lambda|}$$

for all $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : \ |\arg \lambda| \leqslant \pi - \theta\} \cup \{0\}$. In addition

$$\|(\lambda + A_\nu)^{-1} - (\lambda + A_\mu)^{-1}\| \leqslant C\|A_\nu^{-1} - A_\mu^{-1}\| \tag{11.32}$$

and for any $\alpha \in [0,1]$, $\nu \in [0,1]$, $i = 0,1$,

$$\|A_\nu^i e^{-A_\nu t} - A_\mu^i e^{-A_\mu t}\| \leqslant M t^{-i-\alpha}\|A_\nu^{-1} - A_\mu^{-1}\|^\alpha. \tag{11.33}$$

Theorem 11.14 ([13]). *For any $0 < \epsilon < 1$ and $\nu \in [0,1]$,*

$$\|B_\nu e^{-A_\nu t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant M t^{-1+\frac{\epsilon}{2}}.$$

Furthermore, given $0 < \epsilon < 1$ and $\alpha < \frac{\epsilon}{2(1-\epsilon)}$ we have that $\beta = 1 + \alpha(1-\epsilon) - \frac{\epsilon}{2} < 1$ and

$$\|B_\nu e^{-A_\nu t} - B_\mu e^{-A_\mu t}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \leqslant M t^{-\beta}\|A_\nu^{-1} - A_\mu^{-1}\|^{\alpha(1-\epsilon)}$$

for all $\mu, \nu \in [0,1]$.

References

[1] E. A. M. Abreu and A. N. Carvalho, Lower Semicontinuity of Attractors for Parabolic Problems with Dirichlet Boundary Conditons in Varying Domains, *Matemática Contemporânea* **27**, 37-51 (2004).

[2] J.M. Arrieta, A.N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Trans. Am. Math. Soc.* **352** (1999), 285-310.

[3] J. M. Arrieta and A. N. Carvalho, Spectral convergence and nonlinear dynamics of reaction-diffusion equations under perturbations of the domain, *J. Differential Equations* **199**, pp. 143-178, (2004).

[4] J. M. Arrieta, A. N. Carvalho and A. Rodriguez-Bernal, Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds, *Communications in Partial Differential Equations* **25** (1-2), pp. 1-37, (2000).

[5] J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz, Dynamics in dumbbell domains II. The limiting problem. *J. Differential Equations*, **247**, pp 174-202 (2009).

[6] J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz, Dynamics in dumbbell domains III. Continuity of attractors. *J. Differential Equations*, **247**, pp 225-259 (2009).

[7] A. V. Babin and M. I. Vishik, *Attractors in Evolutionary Equations* Studies in Mathematics and its Applications **25**, North-Holland Publishing Co., Amsterdam, 1992.

[8] S. M. Bruschi, A. N. Carvalho, J. W. Cholewa and T. Dłotko, Uniform Exponential Dychotomy and Continuity of Attractors for Singularly Perturbed Damped Wave Equations, *J. Dyn. Diff. Equ.*, 18 (3) (2006), 767-814.

[9] T. Caraballo, J.A. Langa and J.C. Robinson, Upper semicontinuity of attractors for small random perturbations of dynamical systems, *Comm. in Partial Differential Equations* 23 (1998), 1557-1581.

[10] T. Caraballo and J. A. Langa, On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **10** 491–513 (2003)

[11] V. L. Carbone, A. N. Carvalho, and K. Schiabel-Silva, Continuity of Attractors for parabolic problems with localized large diffusion, *Nonlinear Analysis: Theory, Methods and Applications*, 68 (3) (2008), 515-535.

[12] A. N. Carvalho, J. W. Cholewa and Tomasz Dlotko, 'Singular semigroups under singular perturbations' Preprint.

[13] A.N. Carvalho, T. Dlotko, Dynamics of the viscous Cahn-Hilliard equation, *J. Math. Anal. Appl.* **344** (2008), 703-725.

[14] A. N. Carvalho and J. A. Langa, The existence and continuity of stable and unstable manifolds for semilinear problems under non-autonomous perturbation in Banach spaces, *J. Differential Equations*, **233** (2) (2007) 622-653.

[15] A. N. Carvalho and J. A. Langa, An extension of the concept of gradient systems which is stable under perturbation, *J. Differential Equations*, 246 (7), (2009) 26462668.

[16] A. N. Carvalho, J. A. Langa and J. C. Robinson, Lower semicontinuity of attractors for non-autonomous dynamical systems, *Ergodic Th. of Dyn. Systems*, doi:10.1017/S0143385708000850.

[17] A. N. Carvalho, J. A. Langa and J. C. Robinson, On the continuity of pullback attractors for evolutions processes, *Nonlinear Analysis TMA*, doi:10.1016/j.na.2009.01.016.

[18] A. N. Carvalho, J. A. Langa, J. C. Robinson and A. Suárez, Characterization of non-autonomous attractors of a perturbed gradient system, *J. Differential Equations* **236** (2) (2007), 570-603.

[19] A. N. Carvalho and S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems, *Numerical Functional Analysis and Optimization*, **27** (7-8) 785–829 (2006).

[20] V. V. Chepyzhov and M. I. Vishik (2002) *Attractors for Equations of Mathematical Physics* (Providence, AMS Colloquium Publications vol. 49, A.M.S).

[21] C. Conley, The gradient structure of a flow I, Ergodic Theory Dynam. Systems 8* (1988), Charles Conley Memorial Issue, 11–26, 9

[22] H. Crauel, A. Debussche and F. Flandoli, Random attractors. *J. Dyn. Diff. Equ.* **9** (1997) pp. 397–341.

[23] M. Efendiev and S. Zelik, The regular attractor for the reaction-diffusion system with a nonlinearity rapidly oscillating in time and its averaging. *Adv. Differential Equations* 8 no. 6, (2003), 673–732.

[24] J. K. Hale *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs Number 25 (American Mathematical Society, Providence, RI) (1988).

[25] J. K. Hale and H. Koçak, *Dynamics and Bifurcations*, Texts in Applied Mathematics **03**, Springer Verlag, New York 1991.

[26] J. K. Hale and G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, *Ann. Mat. Pura Appl.* **154** (4) (1989), 281–326.

[27] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, 1981.

[28] D. Henry, *Evolution Equations in Banach Spaces*, Dan Henry's Manuscripts, Publicações do IME-USP, ISBN 85-7314-033-X, http://www.ime.usp.br/map/dhenry/ (2006).

[29] N. Ju, D. Small and S. Wiggins, Existence and computation of hyperbolic trajectories of periodically time dependent vector fields and their approximations, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (6) (2003), 1449–1457 .

[30] P.E. Kloeden and B. Schmalfuß (1998), Asymptotic behaviour of non-autonomous difference inclusions. *Systems and Control Letters*, **33**: 275–280, 1998.

[31] O. A. Ladyzhenskaya, *Attractors for semigroups and evolution equations*, Leizioni Lincee, Cambridge Univ. Press, Campridge, UK (1991)

[32] J.A. Langa, J.C. Robinson and A. Suárez, Stability, instability, and bifurcation phenomena in non-autonomous differential equations. *Nonlinearity* 15 (2002), no. 3, 887–903.

[33] J.A. Langa, J.C. Robinson, A. Suárez and A. Vidal-López, The stability of attractors for non-autonomous perturbations of gradient-like systems, *J. Differential Equations* **234** (2) (2007), 607-625.

[34] J.A. Langa, J.C. Robinson, A. Rodríguez-Bernal, A. Suárez, A. Vidal-López, Existence and nonexistence of unbounded forwards attractor for a class of non-autonomous reaction diffusion equations. Discrete Contin. Dyn. Syst. 18 (2007), no. 2-3, 483-497.

[35] N. Malhotra and S. Wiggins, Geometric structures, lobe dynamics, and Lagrangian transport in flows with aperiodic time-dependence, with applications to Rossby wave flow. *J. Nonlinear Sci.* **8** (1998), 401–456.

[36] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Pres, Cabridge UK (2001).

[37] A. Rodríguez-Bernal, Localized spatial homogenization and large diffusion, SIAM Journal of Mathematical Analysis 29 (1998) 1361-1380.

[38] Temam, R. *Infinite dimensional dynamical systems in mechanics and physics*. New York: Springer.

[39] M.I. Vishik and S.V. Zelik, Regular attractors and their nonautonomous perturbations, preprint.

[40] Y. Yi. A generalized integral manifold theorem. *J. Differential Equations* **102** (1993), 153–187.

(J. M. Arrieta) Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

(A. N. Carvalho) Instituto de Ciências Matemáticas e de Computaçao, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil

(J. A. Langa) Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla Spain

(A. Rodriguez-Bernal) Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2009

1. DESIGN OF CODE DIVISION MULTIPLE ACCESS FILTERS USING GLOBAL OPTIMIZATION TECHNIQUES*,* B. Ivorra, B. Mohammadi, and A. M.Ramos

2. DYNAMICS IN DUMBBELL DOMAINS II. THE LIMITING PROBLEM, J.M. Arrieta, A. N. Carvalho and G. Lozada-Cruz

3. DYNAMICS IN DUMBBELL DOMAINS III. CONTINUITY OF ATTRACTORS , J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz

4. GEOMETRIC VERSUS SPECTRAL CONVERGENCE FOR THE NEUMANN LAPLACIAN UNDER EXTERIOR PERTURBATIONS OF THE DOMAIN , J. M. Arrieta and D. Krejcirik

5. ON THE MODELLING AND SIMULATION OF HIGH PRESSURE PROCESSES AND INACTIVATION OF ENZYMES IN FOOD ENGINEERING, J.A. Infante, B. Ivorra, Á.M. Ramos and J.M. Rey

6. CHARACTERIZATION OF THE MARTIAN SURFACE LAYER, G.Martínez, F. Valero, L. Vazquez.

7. CHARACTERIZATION OF THE MARTIAN CONVECTIVE BOUNDARY LAYER, G.Martínez, F. Valero, L. Vazquez.

8. INFINITE RESONANT SOLUTIONS AND TURNING POINTS IN A PROBLEM WITH UNBOUNDED BIFURCATION, J.M. Arrieta, R.Pardo, A.Rodríguez—Bernal

9. CASCADES OF HOPF BIFURCATIONS FROM BOUNDARY DELAY, J.M. Arrieta, N. Cónsul, S. Oliva

10. QUENCHING PHENOMENA FOR A NON-LOCAL DIFFUSION EQUATION WITH A SINGULAR ABSORPTION, R. Ferreira

11. WELL-POSEDNESS OF THE EINSTEIN–EULER SYSTEM IN ASYMPTOTICALLY FLAT SPACETIMES: THE CONSTRAINT EQUATIONS, U. Brauer and L. Karp

12. WELL-POSEDNESS OF THE EINSTEIN–EULER SYSTEM IN ASYMPTOTICALLY FLAT SPACETIMES: THE EVOLUTION EQUATIONS, U. Brauer and L. Karp

13. EXTREMAL EQUILIBRIA FOR MONOTONE SEMIGROUPS IN ORDERED SPACES WITH APPLICATIONS TO EVOLUTIONARY EQUATIONS, J. W. Cholewa and A. Rodriguez-Bernal

14. ASYMPTOTIC BEHAVIOUR OF A PARABOLIC PROBLEM WITH TERMS CONCENTRATED IN THE BOUNDARY, A. Jiménez-Casas and A. Rodriguez-Bernal

15. ASYMPTOTIC BEHAVIOR OF REACTION DIFFUSION EQUATIONS IN WEIGHTED SOBOLEV SPACES, J.M. Arrieta, N. Moya and A. Rodriguez-Bernal

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2010

1. CONTINUITY OF DYNAMICAL STRUCTURES FOR NON-AUTONOMOUS EVOLUTION EQUATIONS UNDER SINGULAR PERTURBATIONS, J. Arrieta, A. N. Carvalho, J. Langa and A. Rodríguez-Bernal