Asymptotic behaviour of a parabolic problem with terms concentrated in the boundary *

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Abstract

We analyze the asymptotic behavior of the attractors of a parabolic problem when some reaction and potential terms are concentrated in a neighborhood of a portion Γ of the boundary and this neighborhood shrinks to Γ as a parameter ε goes to zero.

We prove that this family of attractors is upper continuous at $\varepsilon = 0$.

1 Introduction

Let Ω be an open bounded smooth set in $I\!R^N$ with a C^2 boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be a smooth subset of the boundary, isolated from the rest of the boundary, that is, $\text{dist}(\Gamma, \partial\Omega \setminus \Gamma) > 0$.

Define the strip of width ε and base Γ as

$$\omega_\varepsilon = \{x - \sigma\vec{n}(x),\ x \in \Gamma,\ \sigma \in [0, \varepsilon)\}$$

for sufficiently small ε, say $0 \leq \varepsilon \leq \varepsilon_0$, where $\vec{n}(x)$ denotes the outward normal vector. We note that for small ε, the set ω_ε is a neighborhood of Γ in $\bar{\Omega}$, that collapses to the boundary when the parameter ε goes to zero.

We are interested in the behavior, for small ε, of the solutions of the nonlinear parabolic problem

$$(P_\varepsilon) \equiv \begin{cases} u_t^\varepsilon + A_\varepsilon u^\varepsilon + \mu u^\varepsilon = F_\varepsilon(x, u^\varepsilon) & \text{in } (0,T) \times \Omega \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma \\ u^\varepsilon(0) = u_0 \in H^1(\Omega) \end{cases} \tag{1.1}$$

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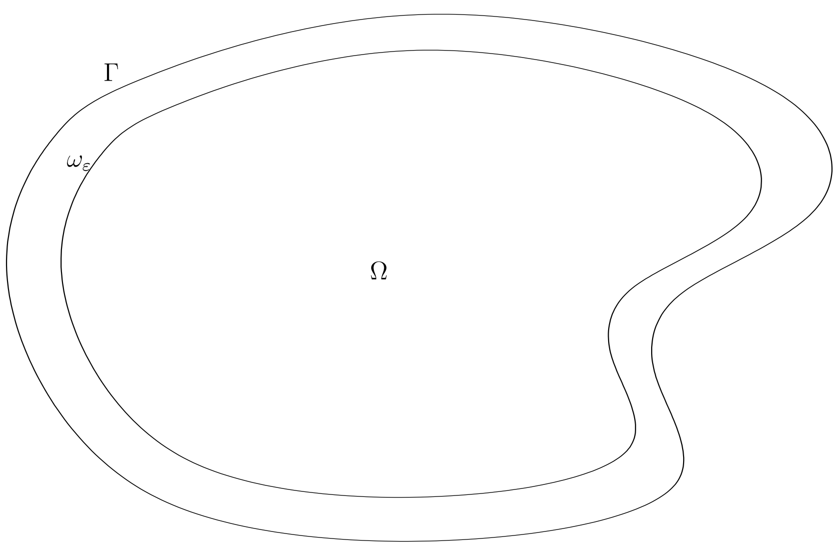

Figure 1: The set ω_ε

where $A_\varepsilon u^\varepsilon = -\Delta u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u_\varepsilon$, $\mu > 0$ is such that the elliptic problem associated is positive and

$$F_\varepsilon(x, u^\varepsilon) = f_0(x, u^\varepsilon) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_0(x, u^\varepsilon),$$

where we denote by $\mathcal{X}_{\omega_\varepsilon}$ the characteristic function of the set ω_ε. Thus the potential functions and the effective reaction are "concentrated" in ω_ε. Note that we assume that both f_0 and g_0 are defined on $\overline{\Omega} \times I\!R$.

We are interested in the behavior, for small ε, of the attractor of this parabolic problem.

Below we will assume several hypotheses that imply

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to \mathcal{X}_\Gamma V_0$$

for some function V_0 defined on Γ and

$$F_\varepsilon(x, u) \to F_0(x, u) = f_0(x, u) + \mathcal{X}_\Gamma g_0(x, u)$$

in "some sense" (see [4]). So, we consider the limit parabolic problem given by

$$(P_0) \equiv \begin{cases} u_t - \Delta u + \mu u = f_0(x, u) & \text{in } (0, T) \times \Omega \\ \frac{\partial u}{\partial n} + V_0 u = g_0(x, u) & \text{on } \Gamma \\ u(0) = u_0 \in H^1(\Omega). & \end{cases} \tag{1.2}$$

Our goal is to prove that the family of sets $\mathcal{A}_\varepsilon$ global attractor of (P_ε) is upper semicontinuous at $\varepsilon = 0$ in $H^1(\Omega)$, that is: $dist_{H^1(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \mapsto 0$, if $\varepsilon \mapsto 0$ with

$$dist_{H^1(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) := \sup_{u^\varepsilon \in \mathcal{A}_\varepsilon} \inf_{u^0 \in \mathcal{A}_0} \{\|u^\varepsilon - u^0\|_{H^1(\Omega)}\}$$

where $\mathcal{A}_0$ is the global attractor associated to the limit problem (1.2).

In order to prove this result about the global attractor for the parabolic problem, we use some previous results about the concentrating integral and the elliptic problem associated to the parabolic problem, see [4, 5].

2 Upper Semicontinuity of Attractors

We consider the family of parabolic problems (1.1), for $\varepsilon \in (0, \varepsilon_0]$

$$(P_\varepsilon) \equiv \begin{cases} u_t^\varepsilon + A_\varepsilon u^\varepsilon + \mu u^\varepsilon = F_\varepsilon(x, u^\varepsilon) & \text{in } (0,T)\times\Omega \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma \\ u^\varepsilon(0) = u_0 \in H^1(\Omega) \end{cases}$$

where $A_\varepsilon u^\varepsilon = -\Delta u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon u_\varepsilon$, and $F_\varepsilon(x, u^\varepsilon) = f_0(x, u^\varepsilon) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} g_0(x, u^\varepsilon)$.

Throughout this section we will assume that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^\rho \le K_1$$

with $\rho > N-1$, and K_1 a positive constant independent of ε, and that there exists a function $V_0 \in L^\rho(\Gamma)$ such that for any smooth function φ, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon \varphi = \int_\Gamma V_0 \varphi.$$

Next, we consider the parabolic problem given by

$$(P_0) \equiv \begin{cases} u_t - \Delta u + \mu u = f_0(x, u) & \text{in } (0,T)\times\Omega \\ \frac{\partial u}{\partial n} + V_0 u = g_0(x, u) & \text{on } \Gamma \\ u(0) = u_0 \in H^1(\Omega). \end{cases}$$

We assume also the the following conditions for the nonlinearity functions f_0, g_0:
Growth conditions (G) : $f_0, g_0 : \overline{\Omega} \times I\!R \mapsto I\!R, \varepsilon \in [0, \varepsilon_0]$ are locally Lipschitz uniformly in $x \in \Omega$ and $x \in \overline{\Omega}$ respectively and satisfy the following growth conditions:

If $N > 2$ we assume that

$$|j(x,u) - j(x,v)| \le c|u-v|(|u|^{\sigma-1} + |v|^{\sigma-1} + 1)$$

with $j = f_0$ or $j = g_0$ and with exponents σ_f and σ_g respectively, such that

$$\sigma_f \le \frac{N+2}{N-2} \quad \text{and} \quad \sigma_g \le \frac{N}{N-2}.$$

If $N = 2$ we assume that for every $\eta > 0$ there exists $c_\eta > 0$ such that

$$|j(x,u) - j(x,v)| \le c_\eta |u-v|(e^{\eta|u|^2} + e^{\eta|v|^2}).$$

These conditions imply the local existence and uniqueness of solutions of (1.1) and (1.2), see J.Arrieta, A. Rodriguez-Bernal et al.[2].

We assume also the following conditions which ensure that local solutions of the nonlinear parabolic problems (1.1) and (1.2) are globally defined and we have well defined semigroups in $H^1(\Omega)$,

$$T_\varepsilon(t)u_0 = u^\varepsilon(t, x; u_0), \qquad 0 \le \varepsilon \le \varepsilon_0,$$

see [3]. Note that the nonlinear semigroups are given by the variation of constants formula

$$T_\varepsilon(t, u_0) = e^{-A_\varepsilon^* t}u_0 + \int_0^t e^{-A_\varepsilon^*(t-s)} F_\varepsilon(\cdot, T_\varepsilon(s, u_0))ds$$

with $A_\varepsilon^* = A_\varepsilon + \mu I$ and $\varepsilon \in [0, \varepsilon_0)$, see [7].

Sign conditions (S) Assume in addition that there exist $C \in L^p(\Omega), 0 \le D \in L^p(\Omega), p > \frac{N}{2}$ and $E \in L^q(\Omega), 0 \le F \in L^q(\Omega), q > N-1$ such that

$$sf_0(x, s) \le C(x)s^2 + D(x)|s|, \quad x \in \Omega, s \in I\!R, \quad 0 \le \varepsilon \le \varepsilon_0$$

and

$$sg_0(x, s) \le E(x)s^2 + F(x)|s|, \quad x \in \overline{\Omega}, s \in I\!R, \quad 0 \le \varepsilon \le \varepsilon_0.$$

Moreover we assume there exist positive constants K_i, $i = 2, 3$ independent of ε such that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |E|^\rho \le K_2, \qquad \text{with } \rho > N-1,$$

and

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |F|^r \le K_3, \qquad \text{with } r > max\{1, \frac{2(N-1)}{N}\}.$$

Dissipative condition (D) Finally we assume the first eigenvalue, λ_1, of the following problem is positive

$$(P_0^1) \equiv \begin{cases} -\Delta\varphi - C\varphi + \mu\varphi = \lambda_1\varphi & \text{in } \Omega \\ \frac{\partial\varphi}{\partial n} + V_0\varphi = E\varphi & \text{on } \Gamma. \end{cases} \tag{2.1}$$

With these assumptions our goal is to prove the upper semicontinuity of the family of global attractors. In order to prove this, we use the previous result for the elliptic problem (see J.Arrieta, A. Jimenez-Casas, A.Rodriguez-Bernal [4]), and the following lemmas.

Lemma 2.1 *Under the above hypotheses on V_ε, f_0, g_0, for sufficiently small $0 \le \varepsilon$, problems (1.1) and (1.2) have global attractors $\mathcal{A}_\varepsilon$.*

Moreover, there exists $R > 0$ independent of $\varepsilon \ge 0$, such that

$$\sup_{0\le\varepsilon\le\varepsilon_0} \sup_{u\in\mathcal{A}_\varepsilon} \|u\|_{L^\infty(\Omega)} \le R.$$

Proof We denote by λ_1^ε the first eigenvalue of the following elliptic problem

$$\begin{cases} -\Delta\phi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}(V_\varepsilon - E)\phi^\varepsilon + (\mu - C)\phi^\varepsilon = \lambda_1^\varepsilon\phi \text{ on } \Omega \\ \frac{\partial\phi^\varepsilon}{\partial n} = 0 \text{ in } \Gamma. \end{cases}$$

By the spectral convergence obtained in [4], we have $\lambda_1^\varepsilon \mapsto \lambda_1$ with λ_1 the first eigenvalue of the elliptic limit problem (2.1). Hence for small enough ε_0 we have $\lambda_1^\varepsilon > 0$ for every $0 \le \varepsilon \le \varepsilon_0$.

We split the proof in several steps.

Step 1: Since, $\lambda_1^\varepsilon > 0$, for $0 \le \varepsilon \le \varepsilon_0$, following the arguments in [3], see also [9], we prove that

$$\lim sup_{t\to\infty}|u^\varepsilon(t, x; u_0)| \le |\Phi^\varepsilon(x)|$$

uniformly in $x \in \overline{\Omega}$ and for $u_0 \in B$ in a bounded set B in $H^1(\Omega)$, where $\Phi^\varepsilon(x)$ is the unique solution of

$$\begin{cases} -\Delta\Phi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}(V_\varepsilon - E)\Phi^\varepsilon + (\mu - C)\Phi^\varepsilon = D + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}F \text{ on } \Omega \\ \frac{\partial\Phi^\varepsilon}{\partial n} = 0 \text{ in } \Gamma. \end{cases}$$

Step 2: From the convergence results for elliptic problems in [4], we prove that $\Phi^\varepsilon(x) \to \Phi^0(x)$, as $\varepsilon \to 0$, in $\mathcal{C}^\beta(\bar{\Omega})$, for some $\beta > 0$, where $\Phi^0(x)$ the unique solution of the following problem

$$\begin{cases} -\Delta\Phi^0 - (C - \mu)\Phi^0 = D \text{ in } \Omega \\ \frac{\partial\Phi^0}{\partial n} + V_0\Phi^0 = E\Phi^0 + F \text{ in } \Gamma. \end{cases}$$

Thus, from the smoothing effect of the equations and the results in [6] we get that problems (1.1) and (1.2) have global compact attractors $\mathcal{A}_\varepsilon$ in $H^1(\Omega)$, for $0 \le \varepsilon \le \varepsilon_0$. Also, there exists R independent of $\varepsilon > 0$ and ε_0 enough small, such that

$$sup_{0\le\varepsilon\le\varepsilon_0} sup_{u\in\mathcal{A}_\varepsilon}\|u\|_{L^\infty(\Omega)} \le R.\square$$

With this and the variation of constants formula we get

Lemma 2.2 *Under the above hypotheses on V_ε, f_0, g_0 we have that, there exists $R > 0$ such that*

$$\sup_{0\le\varepsilon\le\varepsilon_0}\sup_{u\in\mathcal{A}_\varepsilon}\|u\|_{H^1(\Omega)\cap L^\infty(\Omega)} \le R.$$

In particular, $\mathcal{A}_0$ attracts $\cup_{\varepsilon\in(0,\varepsilon_0)}\mathcal{A}_\varepsilon$ in $H^1(\Omega)$.

The following Lemma 2.3 proved in [8] shows several technical results on the behavior of concentrating function as $\varepsilon \mapsto 0$. This will allow us to prove the convergence of the nonlinearities F_ε given by the Lemma 2.4. Note that below we make use of the intermediate sobolev spaces $H^s(\Omega)$ and their dual spaces which we denote $H^{-s}(\Omega)$ for $\frac{1}{2} \le s \le 1$, that is $H^{-s}(\Omega) \equiv (H^s(\Omega))'$.

Lemma 2.3 *Under the above hypotheses, if $\|v\|_{H^1(\Omega)\cap L^\infty(\Omega)} \leq R$, then we have that:*
i) For any $\frac{1}{2} < s \leq 1$ there exists $M(R)$ a positive constant independent of ε such that for any smooth function φ up to the boundary of Ω,

$$|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_0(v)\varphi| \leq M(R)\|\varphi\|_{H^s(\Omega)}.$$

ii) There exists $M(\varepsilon, R) \mapsto 0$ if $\varepsilon \mapsto 0$ such that for any smooth function φ up to the boundary of Ω,

$$|\frac{1}{\varepsilon}\int_{\omega\varepsilon} g_0(v)\varphi - \int_\Gamma g_0(v)\varphi| \leq M(\varepsilon, R)\|\varphi\|_{H^1(\Omega)}.$$

Proof The proof of this Lemma can be found in [8].$\square$

With this, we get the following result that states the convergence of the nonlinear terms of the problems.

Lemma 2.4 *Under the above hypotheses we have that for any $\frac{1}{2} < s < 1$:*
i) There exists $C > 0$ independent of $\varepsilon > 0$ such that

$$\sup_{v\in\mathcal{A}_\varepsilon}\{\|F_\varepsilon(v)\|_{H^{-s}(\Omega)}, \|F_0(v)\|_{H^{-s}(\Omega)}\} \leq C.$$

ii) There exists $M(\varepsilon)$ with $M(\varepsilon) \mapsto 0$ if $\varepsilon \mapsto 0$, such that

$$\sup_{v\in\mathcal{A}_\varepsilon}\|F_\varepsilon(v) - F_0(v)\|_{H^{-s}(\Omega)} \leq M(\varepsilon).$$

Proof Part i) follows from Lemmas 2.2 and 2.3 i). For part ii), note that from Lemmas 2.1 and 2.4, we obtain that:

$$\sup_{v\in\mathcal{A}_\varepsilon}\|F_\varepsilon(v) - F_0(v)\|_{H^{-1}(\Omega)} \leq M(\varepsilon) \to 0$$

as $\varepsilon \to 0$.

Now, fix $\frac{1}{2} < s_0 < 1$. Then for any s such that $-\frac{1}{2} < -s_0 < -s < -1$, by interpolation

$$\|F_\varepsilon(v) - F_0(v)\|_{H^{-s}(\Omega)} \leq \|F_\varepsilon(v) - F_0(v)\|^\theta_{H^{-s_0}(\Omega)}\|F_\varepsilon(v) - F_0(v)\|^{1-\theta}_{H^{-1}(\Omega)}$$

for some $0 < \theta < 1$. Again by Lemmas 2.2 and 2.3 i), the first term in the right hand side above is bounded while the second goes to zero, both uniformly for $v \in \mathcal{A}_\varepsilon$, and we conclude.$\square$

On the other hand, from the spectral convergence of the linear operators we get, see [1] for a similar result.

Lemma 2.5 *Under the above hypotheses, let $\frac{1}{2} < s < 1$. Then, there exist $\alpha \in (\frac{1+s}{2}, 1)$ and a function $C_0(\varepsilon) \geq 0$ with $C_0(\varepsilon) \mapsto 0$ if $\varepsilon \mapsto 0$, such that for all $h \in H^{-s}(\Omega)$ we have that*

$$\|e^{-A_\varepsilon^* t}h - e^{-A_0^* t}h\|_{H^1(\Omega)} \leq C_0(\varepsilon)t^{-\alpha}\|h\|_{H^{-s}(\Omega)}, \quad t > 0.$$

With all the above we can then obtain the convergence of the nonlinear semigroups.

Lemma 2.6 *Under the above hypothesis let* $\frac{1}{2} < s_0 < 1$ *and some fixed* $\tau > 0$. *Then, there exists a function* $C(\varepsilon) \geq 0$ *with* $C(\varepsilon) \mapsto 0$ *if* $\varepsilon \mapsto 0$, *such that for* $u_\varepsilon \in \mathcal{A}_\varepsilon$, $\varepsilon \in (0, \varepsilon_0)$,

$$\|T_\varepsilon(t, u_\varepsilon) - T_0(t, u_\varepsilon)\|_{H^1(\Omega)} \leq M(\tau)C(\varepsilon)t^{-\alpha} \quad \text{for } t \in (0, \tau]$$

for some $\alpha \in (\frac{1+s_0}{2}, 1)$.

Proof We consider the nonlinear semigroup given by the variation of constant formula:

$$T_\varepsilon(t, u_\varepsilon) = e^{-A_\varepsilon^* t}u_\varepsilon + \int_0^t e^{-A_\varepsilon^*(t-s)}F_\varepsilon(x, T_\varepsilon(s, u_\varepsilon))ds \tag{2.2}$$

where $A_\varepsilon^* = A_\varepsilon + \mu I$, $\varepsilon \in [0, \varepsilon_0]$ and

$$T_0(t, u_\varepsilon) = e^{-A_0^* t}u_\varepsilon + \int_0^t e^{-A_0^*(t-s)}F_0(x, T_0(s, u_\varepsilon))ds. \tag{2.3}$$

From (2.2) and (2.3), together with the previous results, we will get below that

$$\|T_\varepsilon(t, u_\varepsilon) - T_0(t, u_\varepsilon)\|_{H^1(\Omega)} \leq M^* C(\varepsilon)t^{-\alpha} +$$

$$+ M^* \int_0^t (t-s)^{-\alpha}\|T_\varepsilon(s, u_\varepsilon) - T_0(s, u_\varepsilon)\|_{H^1(\Omega)}ds \tag{2.4}$$

for some M^* depending on τ. Hence, applying the singular Gronwall Lemma, Lemma 7.1.1 in [7], to (2.4), we get the result.

We now split the proof of (2.4) in several steps. In effect, from (2.2) and (2.3) we have that:

$$\|T_\varepsilon(t, u_\varepsilon) - T_0(t, u_\varepsilon)\|_{H^1(\Omega)} \leq \|e^{-A_\varepsilon^* t}u_\varepsilon - e^{-A_0^* t}u_\varepsilon\|_{H^1(\Omega)} +$$

$$+ \int_0^t \|e^{-A_\varepsilon^*(t-s)}F_\varepsilon(x, T_\varepsilon(s, u_\varepsilon)) - e^{-A_0^*(t-s)}F_\varepsilon(x, T_\varepsilon(s, u_\varepsilon))\|_{H^1(\Omega)}ds +$$

$$+ \int_0^t \|e^{-A_0^*(t-s)}[F_\varepsilon(x, T_\varepsilon(s, u_\varepsilon)) - F_0(x, T_\varepsilon(s, u_\varepsilon))]\|_{H^1(\Omega)}ds +$$

$$+ \int_0^t \|e^{-A_0^*(t-s)}[F_0(x, T_\varepsilon(s, u_\varepsilon)) - F_0(x, T_0(s, u_\varepsilon))]\|_{H^1(\Omega)}ds = I_1 + I_2 + I_3 + I_4.$$

Step 1.- From Lemma 2.5 together with Lemma 2.2, we obtain:

$$I_1 = \|e^{-A_\varepsilon^* t}u_\varepsilon - e^{-A_0 t}u_\varepsilon\|_{H^1(\Omega)} \leq C_0(\varepsilon)t^{-\alpha}\|u_\varepsilon\|_{H^{-s_0}(\Omega)} \leq C_0(\varepsilon)t^{-\alpha}K_0$$

with $0 < C_0(\varepsilon) \mapsto 0$ if $\varepsilon \mapsto 0$ and K_0 a positive constant independent of ε.

Step 2.- Again Lemma 2.5 gives

$$I_2 = \int_0^t \|e^{-A_\varepsilon^*(t-s)}F_\varepsilon(x, T_\varepsilon(s, u_\varepsilon)) - e^{-A_0^*(t-s)}F_\varepsilon(x, T_\varepsilon(s, u_\varepsilon))\|_{H^1(\Omega)}ds \leq$$

$$\leq C_0(\varepsilon)\int_0^t (t-s)^{-\alpha}\|F_\varepsilon(x,T_\varepsilon(s,u_\varepsilon))\|_{H^{-s_0}(\Omega)}ds.$$

Now, from Lemma 2.4 and using thet the attractor $\mathcal{A}_\varepsilon$ is invariant for the semigroup T_ε, we obtain a positive constant K_1 independent of ε such that $\|F_\varepsilon(\cdot,T_\varepsilon(s,u_\varepsilon))\|_{H^{-s_0}(\Omega)} \leq K_1$. From this

$$I_2 \leq C_0(\varepsilon)\frac{K_1}{1-\alpha}t^{1-\alpha} \leq C_0(\varepsilon)K_2 t^{-\alpha}$$

since $t \leq \tau$.

Step 3.-

$$I_3 = \int_0^t \|e^{-A_0^*(t-s)}(F_\varepsilon(x,T_\varepsilon(s,u_\varepsilon)) - F_0(x,T_\varepsilon(s,u_\varepsilon))\|_{H^1(\Omega)}ds \leq$$

$$\leq K_2\int_0^t (t-s)^{-\alpha}\|F_\varepsilon((x,T_\varepsilon(s,u_\varepsilon) - F_0(x,T_\varepsilon(s,u_\varepsilon))\|_{H^{-s_0}(\Omega)}ds.$$

Using again Lemma 2.4 and the invariance of the attractor, we obtain that $\|F_\varepsilon(\cdot,T_\varepsilon(s,u_\varepsilon)) - F_0(\cdot,T_\varepsilon(s,u_\varepsilon))\|_{H^{-s_0}(\Omega)} \leq M(\varepsilon)$, with $M(\varepsilon)\mapsto 0$ if $\varepsilon\mapsto 0$ and $I_3 \leq M(\varepsilon)K_3 t^{-\alpha}$, since $t\leq\tau$, with K_3 a positive constant independent of ε and depending on τ.

Step 4.-

$$I_4 = \int_0^t \|e^{-A_0^*(t-s)}(F_0(x,T_\varepsilon(s,u_\varepsilon)) - F_0(x,T_0(s,u_\varepsilon)))\|_{H^1(\Omega)}ds \leq$$

$$\leq K_2\int_0^t (t-s)^{-\alpha}\|F_0(x,T_\varepsilon(s,u_\varepsilon)) - F_0(x,T_0(s,u_\varepsilon))\|_{H^{-s_0}(\Omega)}ds.$$

Now, from the bounds in Lemma 2.2 and the regularity of the nonlinear terms f_0 and g_0 we get that $\|F_0(u)-F_0(v)\|_{H^{-s_0}(\Omega)} \leq L\|u-v\|_{H^1(\Omega)}$ with $L = L(R)$ if the norm of both u and v in $H^1(\Omega)\cap L^\infty(\Omega)$ is bounded by R.

Therefore, we get $I_4 \leq K_2 L\int_0^t (t-s)^{-\alpha}\|T_\varepsilon(s,u_\varepsilon) - T_0(s,u_\varepsilon)\|_{H^1(\Omega)}ds$.

Puting all the estimates above together, we get (2.4) and the proof is complete.□

Theorem 2.7 *Under the above hypothesis about V_ε, f_0, g_0 the family of global attractors of (P_ε), $\mathcal{A}_\varepsilon$, is upper semicontinuous at $\varepsilon = 0$ in $H^1(\Omega)$, that is:*

$$dist_{H^1(\Omega)}(\mathcal{A}_\varepsilon,\mathcal{A}_0)\mapsto 0, \ \ if\ \varepsilon\mapsto 0$$

where

$$dist_{H^1(\Omega)}(\mathcal{A}_\varepsilon,\mathcal{A}_0) := \sup_{u_\varepsilon\in\mathcal{A}_\varepsilon}\inf_{u_0\in\mathcal{A}_0}\{\|u_\varepsilon - u_0\|_{H^1(\Omega)}\}$$

Proof In effect, from Lemma 2.2, $\mathcal{A}_0$ attracts $\cup_{0<\varepsilon\leq\varepsilon_0}\mathcal{A}_\varepsilon$, since the latter is a bounded set in $H^1(\Omega)$. Hence, given $\delta>0$, there exists $\tau=\tau(\delta)$ such that $dist_{H^1}(T_0(\tau)u_\varepsilon,\mathcal{A}_0)\leq\frac{\delta}{2}$ for every $u_\varepsilon\in\mathcal{A}_\varepsilon$ with $\varepsilon\in(0,\varepsilon_0)$.

Next, using that $\mathcal{A}_\varepsilon$ is invariant, given $v_\varepsilon\in\mathcal{A}_\varepsilon$, there exists u_ε such that $T_\varepsilon(\tau)u_\varepsilon = v_\varepsilon$. Therefore,

$$dist_{H^1}(v_\varepsilon,\mathcal{A}_0)\leq\|v_\varepsilon - T_0(\tau)u_\varepsilon\|_{H^1(\Omega)} + dist_{H^1}(T_0(\tau)u_\varepsilon,\mathcal{A}_0).$$

Then from Lemma 2.6 it is clear that if ε is small enough we get

$$\|v_\varepsilon - T_0(\tau)u_\varepsilon\|_{H^1(\Omega)} = \|T_\varepsilon(\tau)u_\varepsilon - T_0(\tau)u_\varepsilon\|_{H^1(\Omega)} \leq \frac{\delta}{2},$$

and we conclude. $\square$

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