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Aníbal Rodríguez-Bernal

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http://www.mat.ucm.es/deptos/ma
e-mail:matemática_aplicada@mat. ucm.es

A singular perturbation in a linear parabolic equation with terms concentrating on the boundary *

Aníbal Rodríguez-Bernal

Departamento de Matemática Aplicada
Universidad Complutense de Madrid,
Madrid 28040, SPAIN
and
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM

Abstract

We analyze the regularity and convergence of the solutions of linear parabolic problems when some reaction and potential terms are concentrated in a neighborhood of a portion Γ of the boundary and this neighborhood shrinks to Γ as a parameter ε goes to zero.

1 Introduction

Let Ω be an open bounded smooth set in $\mathbb{R}^N$ with a C^2 boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be a smooth subset of the boundary, isolated from the rest of the boundary, that is, $\mathrm{dist}(\Gamma, \partial\Omega \setminus \Gamma) > 0$. Define the strip of width ε and base Γ as

$$\omega_\varepsilon = \{x - \sigma\vec{n}(x),\ x \in \Gamma,\ \sigma \in [0, \varepsilon)\}$$

for sufficiently small ε, say $0 < \varepsilon < \varepsilon_0$, where $\vec{n}(x)$ denotes the outward normal vector to Γ. We note that for small ε, the set ω_ε is a neighborhood of Γ in $\overline{\Omega}$, that collapses to Γ when the parameter ε goes to zero.

It was recently proved in [5], in the context of elliptic problems, that boundary potentials for Robin boundary conditions in Γ can be efficiently approximated by concentrating potentials in ω_ε.

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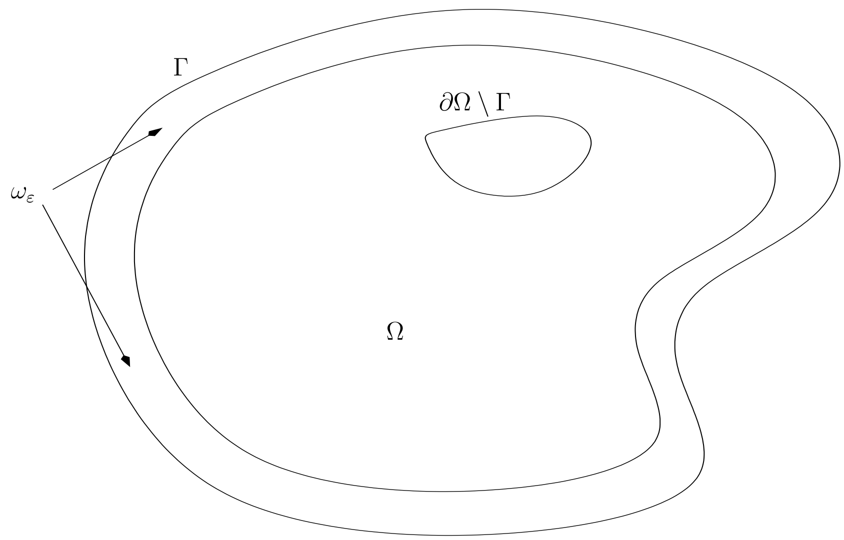

Figure 1: The sets Ω and ω_ε

Hence our goal in this paper is to extend such analysis to linear parabolic equations. To be more precise, we are interested in the behavior, for small ε, of the solutions of the linear parabolic problem

$$\begin{cases} u_t^\varepsilon - \operatorname{div}(a(x)\nabla u^\varepsilon) + c(x)u^\varepsilon &= \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)u^\varepsilon + f(x) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon(x) & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon &= 0 & \text{on } \Gamma \\ \mathcal{B}u^\varepsilon &= 0 & \text{on } \partial\Omega \setminus \Gamma \\ u^\varepsilon(0) &= u_0 & \text{in } \Omega \end{cases}$$

where $a \in C^1(\overline{\Omega})$ with $a(x) \geq a_0 > 0$ in Ω, $c \in C^1(\overline{\Omega})$ and $\mathcal{B}$ denotes the boundary operator in $\partial\Omega \setminus \Gamma$

$$\mathcal{B}u = u, \quad \text{Dirichlet case,} \quad \text{or} \quad \mathcal{B}u = a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u, \quad \text{Robin case,}$$

being $\vec{n}$ the outward normal vector-field to $\partial\Omega \setminus \Gamma$ and $b(x)$ a $C^1(\partial\Omega)$ function and $\mathcal{X}_{\omega_\varepsilon}$ denotes the characteristic function of the set ω_ε.

Note that in this problem some terms are only effective on the region ω_ε which collapses to Γ as $\varepsilon \to 0$.

We will show in this paper that the "limit problem" for the singularly perturbed problem above is given by

$$\begin{cases} u_t - \operatorname{div}(a(x)\nabla u) + c(x)u &= f(x) & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &= V_0(x)u + h_0(x) & \text{on } \Gamma \\ \mathcal{B}u &= 0 & \text{on } \partial\Omega \setminus \Gamma \\ u(0) &= u_0 & \text{in } \Omega \end{cases}$$

where h_0, V_0 are obtained as the limits of the concentrating terms

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \to h_0, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0,$$

in some sense that we now make precise. For this, we have the following definition.

Definition 1.1 *Consider a family of functions $J = \{j_\varepsilon\}_\varepsilon$ in Ω.*
i) The family J is denoted an "L^r–concentrated bounded family" near Γ, if

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |j_\varepsilon|^r \leq C$$

for $1 \leq r < \infty$, or

$$\sup_{x\in\omega_\varepsilon} |j_\varepsilon(x)| \leq C$$

for the case $r = \infty$, and C a positive constant independent of ε.
ii) The family J is an "L^r–concentrated convergent family" if it satisfies that for any smooth function φ in $\overline{\Omega}$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\omega_\varepsilon} j_\varepsilon \varphi = \int_\Gamma j_0 \varphi, \tag{1.1}$$

where $j_0 \in L^r(\Gamma)$ (or a bounded Radon measure on Γ, $j_0 \in \mathcal{M}(\Gamma)$ if $r = 1$). In such a case we write

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} j_\varepsilon \to j_0 \quad cc - L^r.$$

iii) The family J is said to be "L^r–concentrated (sequentially) compact family" if for any sequence in the family there exist a subsequence (that we still denote the same) and a function $j_0 \in L^r(\Gamma)$ (or a bounded Radon measure on Γ, $j_0 \in \mathcal{M}(\Gamma)$ if $r = 1$) such that for any smooth function φ in $\overline{\Omega}$, we have (1.1).

Therefore the results of Lemma 2.2 in [5] can be recast as

Lemma 1.2 *With the notations above, an "L^r–concentrated bounded family" is an "L^r–concentrated (sequentially) compact family".*

Hence, we will assume that

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \to h_0, \qquad \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \text{for some } r > N-1. \tag{1.2}$$

On the other hand, because of it interest in applicatons, we are also interested in dealing with non smooth potentials in Ω and Γ, therefore we consider here the homogeneous problems

$$\left\{\begin{array}{rcll} u_t^\varepsilon - \operatorname{div}(a(x)\nabla u^\varepsilon) + c(x)u^\varepsilon &=& m(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x) u^\varepsilon & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon &=& m_0(x)u^\varepsilon & \text{on } \Gamma \\ \mathcal{B}u^\varepsilon &=& 0 & \text{on } \partial\Omega\setminus\Gamma \\ u^\varepsilon(0) &=& u_0 & \text{in } \Omega \end{array}\right. \tag{1.3}$$

and

$$\left\{\begin{array}{rcll} u_t - \mathrm{div}(a(x)\nabla u) + c(x)u &=& m(x)u & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& (m_0(x) + V_0(x))u & \text{on } \Gamma \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \setminus \Gamma \\ u(0) &=& u_0 & \text{in } \Omega \end{array}\right. \tag{1.4}$$

with $m \in L^p(\Omega)$, $p > N/2$ and $m_0 \in L^r(\Gamma)$, $r > N-1$.

Therefore, our goal is to analyze regularity of solutions of (1.3) and (1.4) and to show that solutions of the former converge to solutions of the latter, assumed that (1.2) holds. More precisely, we are going to prove, among others, the following three main results.

Theorem 1.3 *Assume that m lies in a bounded set in $L^p(\Omega)$, with $p > N/2$, m_0 lies in a bounded set in $L^r(\Gamma)$ and also that the family of potentials V_ε is a L^r–concentrated bounded family, for $r > N-1$, that is*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^r \le C, \quad r > N-1.$$

Then, for any $1 < q < \infty$, the problem (1.3) defines a strongly continuous, order preserving, analytic semigroup, $S_{m,m_0,\varepsilon}(t)$ in the space $H^{2\gamma,q}_{bc}(\Omega)$ for any

$$\gamma \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'}).$$

Moreover the semigroup satisfies the smoothing estimates

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{H^{2\gamma',q}_{bc}(\Omega)} \le \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega)$$

for every $\gamma, \gamma' \in I(q)$, with $\gamma' \ge \gamma$, for some $M_{\gamma',\gamma}$ and $\mu \in I\!R$ independent of m, m_0 and $0 < \varepsilon \le \varepsilon_0$ and $\gamma, \gamma' \in I(q)$. In particular, one has

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{L^\tau(\Omega)} \le \frac{M_{\rho,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \qquad t > 0, \quad u_0 \in L^\rho(\Omega)$$

for $1 \le \rho \le \tau \le \infty$ with $M_{\rho,\tau}$ and μ independent of m, m_0 and $0 < \varepsilon \le \varepsilon_0$.

Finally, for every $u_0 \in H^{2\gamma,q}_{bc}(\Omega)$, with $\gamma \in I(q)$, the function $u(t; u_0) := S_{m,m_0,\varepsilon}(t)u_0$ is a weak solution of (1.3) in the sense that

$$\int_\Omega u_t\varphi + \int_\Omega (a(x)\nabla u\nabla\varphi + (c(x) - m(x))u)\varphi + \int_\Gamma (b(x) - m_0(x))u\varphi = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon(x)u\varphi$$

for all sufficiently smooth φ.

In particular, if $q > N-1$, then there exists $\gamma' \in I(q)$ such that $H^{2\gamma',q}_{bc}(\Omega) \subset C^\beta(\overline{\Omega})$ for some $\beta > 0$ and the solutions of (1.3) become $C^\beta(\overline{\Omega})$ smooth.

Note that in the statement above, $H^{2\gamma,q}_{bc}(\Omega)$ stand for suitable subspaces of the Bessel potential spaces, which are described in Section 2. Also, note that if $V_0 \in L^r(\Gamma)$, for $r > N-1$, with the choice $V_\varepsilon = 0$ and $m_0 + V_0$ replacing m_0, the result above allows to define the semigroup $S_{m,m_0+V_0}(t)$ such that for every $u_0 \in H^{2\gamma,q}_{bc}(\Omega)$, with γ as above, the function $u(t;u_0) := S_{m,m_0+V_0}(t)u_0$ is a weak solution of (1.4). With these notations we have

Theorem 1.4 *Assume*

$$m_\varepsilon \to m \quad \text{in } L^p(\Omega), \quad p > \frac{N}{2},$$
$$m_{0,\varepsilon} \to m_0 \quad \text{in } L^r(\Gamma), \quad r > N-1,$$
$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \text{for some } r > N-1$$

and for any $1 < q < \infty$, consider the semigroups $S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)$ and $S_{m,m_0+V_0}(t)$ as above.

Then for every

$$\gamma, \gamma' \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'}), \quad \gamma' \geq \gamma,$$

and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t) - S_{m,m_0+V_0}(t)\|_{\mathcal{L}(H^{2\gamma,q}_{bc}(\Omega), H^{2\gamma',q}_{bc}(\Omega))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \text{for all} \quad 0 < t \leq T.$$

In particular, if $q > N-1$, then there exists $\gamma' \in I(q)$ such that $H^{2\gamma',q}_{bc}(\Omega)) \subset C^\beta(\overline{\Omega})$ for some $\beta > 0$ and the solutions of (1.3) converge to solutions of (1.4) uniformly in $\overline{\Omega}$.

Finally, about the optimal exponential bound for the semigroups above we have the following

Proposition 1.5 *Assume*

$$m_\varepsilon \to m \quad \text{in } L^p(\Omega), \quad p > \frac{N}{2},$$
$$m_{0,\varepsilon} \to m_0 \quad \text{in } L^r(\Gamma), \quad r > N-1,$$
$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \text{for some } r > N-1$$

and denote by λ_1^ε the first eigenvalue of the following eigenvalue problem

$$\left\{ \begin{array}{rcll} -div(a(x)\nabla\varphi^\varepsilon) + c(x)\varphi^\varepsilon &=& m_\varepsilon(x)\varphi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon + \lambda\varphi^\varepsilon & \text{in } \Omega \\ a(x)\frac{\partial\varphi^\varepsilon}{\partial\vec{n}} + b(x)\varphi^\varepsilon &=& m_{0,\varepsilon}(x)\varphi^\varepsilon & \text{on } \Gamma \\ \mathcal{B}\varphi^\varepsilon &=& 0 & \text{on } \partial\Omega\setminus\Gamma. \end{array} \right.$$

i) We have that

$$\lambda_1^\varepsilon \to \lambda_1^0$$

which is the first eigenvalue of the limit eigenvalue problem

$$\begin{cases} -div(a(x)\nabla\varphi) + c(x)\varphi & = & m(x)\varphi + \lambda\varphi & \text{in } \Omega, \\ a(x)\frac{\partial\varphi}{\partial\vec{n}} + b(x)\varphi & = & (m_0(x) + V_0(x))\varphi & \text{on } \Gamma, \\ \mathcal{B}\varphi & = & 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

ii) For sufficiently small ε *and for any* $-\mu < \lambda_1^0$, *the semigroups* $S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)$ *and* $S_{m,m_0+V_0}(t)$ *defined above satisfy*

$$\|S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)u_0\|_{H^{2\gamma',q}_{bc}(\Omega)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega)$$

$$\|S_{m,m_0+V_0}(t)u_0\|_{H^{2\gamma',q}_{bc}(\Omega)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega)$$

for every $\gamma, \gamma' \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$, *with* $\gamma' \geq \gamma$, *for some* $M_{\gamma',\gamma}$ *independent of* $0 < \varepsilon \leq \varepsilon_0$. *In particular,*

$$\|S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\rho,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \qquad t > 0, \quad u_0 \in L^\rho(\Omega)$$

and

$$\|S_{m,m_0+V_0}(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\rho,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \qquad t > 0, \quad u_0 \in L^\rho(\Omega)$$

with $M_{\rho,\tau}$ *independent of* $0 < \varepsilon \leq \varepsilon_0$.

These results are obtained in Corollary 2.6, Theorem 3.20, Theorem 4.4 and Proposition 4.7 which come our of some general perturbation results on analytic semigroups in scales of Banach spaces, developed in Sections 3 and 4; see Lemma 3.2, Theorem 3.13, Proposition 3.15, Theorem 4.1 and Corollary 4.3. These general results may be applied to many other perturbation problems.

Finally Section 5 contains some further remarks on similar parabolic problems and in particular, on the nonhomogeneous problem.

Some results on the corresponding nonlinear problems have been announced in [7].

2 Functional setting and resolvent estimates for elliptic operators

In this section we analyze the linear homogeneous problems (1.3) and (1.4) by means of resolvent estimates for the associated elliptic operators.

For this, denote by A_0 the operator $A_0u = -\mathrm{div}(a(x)\nabla u) + c(x)u$ with boundary conditions $a(x)\frac{\partial u}{\partial\vec{n}} + b(x)u = 0$ on Γ and $\mathcal{B}u = 0$ on $\partial\Omega \setminus \Gamma$. Note the coefficients a, b, c are C^1–smooth. Also, note that all the analysis below applies in the case the diffusion coefficient is a positive definite matrix instead of a scalar coefficient. We deal with the latter case here only because the notations become simpler.

Choosing $L^q(\Omega)$, for $1 < q < \infty$, as a base space, the unbounded linear operator $A_0 : D(A_0) \subset L^q(\Omega) \to L^q(\Omega)$, with domain $D(A_0) = H^{2,q}_{bc}(\Omega)$, consisting of all functions in $H^{2,q}(\Omega)$ which satisfy all boundary conditions above, generates an analytic semigroup in $L^q(\Omega)$, see [2]. Here and below $H^{s,q}(\Omega)$ denote the Bessel potentials spaces which, for integer s, coincide with the usual Sobolev spaces.

Using the complex interpolation–extrapolation procedure, one can construct the scale of Banach spaces associated to this operator, which will be denoted $H^{2\alpha,q}_{bc}(\Omega)$ for $\alpha \in [-1, 1]$, which are closed subspaces of $H^{2\alpha,q}(\Omega)$ incorporating some of the boundary conditions. In particular, we have $H^{0,q}_{bc}(\Omega) = L^q(\Omega)$, and

$$H^{1,q}_{bc}(\Omega) = \begin{cases} \{u \in W^{1,q}(\Omega) : \ u = 0 \text{ in } \partial\Omega \setminus \Gamma\} & \text{for } \mathcal{B} \text{ Dirichlet} \\ W^{1,q}(\Omega) & \text{for } \mathcal{B} \text{ Robin.} \end{cases}$$

Recall that Bessel spaces have the sharp embeddings

$$H^{s,q}(\Omega) \subset \begin{cases} L^r(\Omega), \ s - \frac{N}{q} \geq -\frac{N}{r}, \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Omega), \ \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\bar{\Omega}) & \text{if } s - \frac{N}{q} > \eta > 0 \end{cases}$$

with continuous embeddings, see [1]. This embeddings are known to be optimal.

Also, if T denotes the trace operator, then for $s > \frac{1}{q}$, T is well defined on $H^{s,q}(\Omega)$ and

$$H^{s,q}(\Omega) \xrightarrow{T} \begin{cases} L^r(\Gamma), \ s - \frac{N}{q} \geq -\frac{N-1}{r}, \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Gamma), \ \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\Gamma) & \text{if } s - \frac{N}{q} > \eta > 0 \end{cases}$$

see [1].

Note that the scale with negative exponents satisfies $H^{-2\alpha,q}_{bc}(\Omega) = (H^{2\alpha,q'}_{bc}(\Omega))'$, for $0 < \alpha < 1$. Moreover, we have $H^{-2\alpha,q}(\Omega) = (H^{2\alpha,q'}(\Omega))'$ and $H^{-2\alpha,q}(\Omega) \hookrightarrow H^{-2\alpha,q}_{bc}(\Omega)$. See [2] for details. Using this it is easy to obtain that for $s > 0$ we have

$$H^{-s,q}(\Omega) \supset \begin{cases} L^r(\Omega), \ -s - \frac{N}{q} \leq -\frac{N}{r}, \ 1 < r \leq \infty, & \text{if } s - \frac{N}{q'} < 0 \\ L^r(\Omega), \ \ 1 < r \leq \infty, & \text{if } s - \frac{N}{q'} = 0 \\ \mathcal{M}(\Omega) & \text{if } s - \frac{N}{q'} > 0. \end{cases}$$

Then, the operator $-A_0$ or, more precisely, a suitable realization of it, generates an analytic semigroup, $S_0(t)$, in each space of the scale $H^{2\alpha,q}_{bc}(\Omega)$, $\alpha \in [-1, 1]$. This semigroup is order preserving and satisfies the smoothing estimates

$$\|S_0(t)u_0\|_{H^{2\alpha,q}_{bc}(\Omega)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\beta,q}_{bc}(\Omega) \tag{2.1}$$

for $1 \geq \alpha \geq \beta \geq -1$ and some $\mu \in I\!R$. In particular, one has

$$\|S_0(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\tau,\rho}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \qquad t > 0, \quad u_0 \in L^\rho(\Omega) \tag{2.2}$$

for $1 \leq \rho \leq \tau \leq \infty$. For any u_0 in $H^{2\beta,q}_{bc}(\Omega)$ or $L^\rho(\Omega)$, the function $u(t;u_0) := S_0(t)u_0$, $t > 0$, is a classical solution of (1.3) for $V_\varepsilon = m = m_0 = 0$. The reader is referred to [2] and references therein, for further properties of this scale of spaces and semigroups.

Note that this construction applies to much more general elliptic operators than above. Also, in the construction above the regularity of the coefficients, plays a fundamental role; see [2].

Therefore we consider now nonsmooth perturbations of the operator A_0. More precisely we consider a nonsmooth potential $m(x)$ in Ω, a nonsmooth perturbation, $m_0(x)$ of the boundary coefficient $b(x)$ in Γ as well as a family of concentrated perturbations in Γ

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x).$$

In order to treat all perturbations in a unified form, we define for $0 < \varepsilon \leq \varepsilon_0$,

$$< P_\varepsilon u, \varphi >= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon u \varphi, \tag{2.3}$$

$$< Q_0 u, \varphi >= \int_\Omega m u \varphi, \qquad < R_0 u, \varphi >= \int_\Gamma m_0 u \varphi \tag{2.4}$$

for suitable u and φ. Then we have

Lemma 2.1 *Assume that m lies in a bounded set in $L^p(\Omega)$, m_0 lies in a bounded set in $L^r(\Gamma)$ and also that the family of potentials V_ε is a L^r–concentrated bounded family, that is*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C.$$

Then
i) for $s, \sigma \geq 0$ and

$$s + \sigma > \frac{N}{p} \tag{2.5}$$

we have

$$Q_0 \in \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega)).$$

and is a bounded family in that space.
ii) for $s > 1/q$, $\sigma > 1/q'$ and

$$s + \sigma > 1 + \frac{N-1}{r} \tag{2.6}$$

satisfy

$$P_\varepsilon, R_0 \in \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))$$

and are bounded families in that space.

Proof. i) Note that for every $u \in H^{s,q}(\Omega)$ and $\varphi \in H^{\sigma,q'}(\Omega)$ we have

$$|\int_\Omega m u \varphi| \leq (\int_\Omega |m|^p)^{\frac{1}{p}} (\int_\Omega |u|^\rho)^{\frac{1}{\rho}} (\int_\Omega |\varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{p}+\frac{1}{\rho}+\frac{1}{\tau}=1$. Using the sharp embedding of the Bessel spaces, we have

$$|\int_\Omega mu\varphi| \leq C\|u\|_{H^{s,q}(\Omega)}\|\varphi\|_{H^{\sigma,q'}(\Omega)}$$

provided ρ,τ are such that $s-\frac{N}{q}\geq -\frac{N}{\rho}$, and $\sigma-\frac{N}{q'}\geq -\frac{N}{\tau}$. These conditions can be met because of (2.5).
ii) Now note that for every $u\in H^{s,q}(\Omega)$ and $\varphi\in H^{\sigma,q'}(\Omega)$ we have

$$|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon u\varphi| \leq (\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^r)^{\frac{1}{r}}(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u|^m)^{\frac{1}{m}}(\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^n)^{\frac{1}{n}}$$

where $\frac{1}{r}+\frac{1}{m}+\frac{1}{n}=1$. Using Lemma 2.1 in [5] we have

$$|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon u\varphi| \leq C\|u\|_{H^{s,q}(\Omega)}\|\varphi\|_{H^{\sigma,q'}(\Omega)}$$

provided m,n are such that $s-\frac{N}{q}\geq -\frac{N-1}{m}$, with $s>\frac{1}{q}$, and $\sigma-\frac{N}{q'}\geq -\frac{N-1}{n}$, with $\sigma>\frac{1}{q'}$. These conditions can be met because of (2.6). The case of R_0 is entirely similar, using integrals on Γ. ■

Then, we have the following result, which is, in particular, an improvement of Theorem 3.1 in [5].

Theorem 2.2 *Assume that m lies in a bounded set in $L^p(\Omega)$, with $p>N/2$, m_0 lies in a bounded set in $L^r(\Gamma)$ and also that the family of potentials V_ε is a L^r–concentrated bounded family, for $r>N-1$, that is*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|V_\varepsilon|^r\leq C, \quad r>N-1.$$

Then, for any $1<q<\infty$, there exists some $\omega_0>0$ independent of m, m_0 and ε, such that for any $Re(\lambda)\geq\omega_0$ and any $\sigma\in(\frac{1}{q'},2-\frac{1}{q})$ the elliptic operator $A_0+\lambda I-(P_\varepsilon+Q_0+R_0)$, between $H^{2-\sigma,q}_{bc}(\Omega)$ and $H^{-\sigma,q}_{bc}(\Omega)$, is invertible and

$$\|(A_0+\lambda I-(P_\varepsilon+Q_0+R_0))^{-1}\|_{\mathcal{L}(H^{-\sigma,q}_{bc}(\Omega),H^{-\sigma,q}_{bc}(\Omega))}\leq \frac{C}{|\lambda|}, \qquad Re(\lambda)\geq\omega_0 \tag{2.7}$$

and

$$\|(A_0+\lambda I-(P_\varepsilon+Q_0+R_0))^{-1}\|_{\mathcal{L}(H^{-\sigma,q}_{bc}(\Omega),H^{2-\sigma,q}_{bc}(\Omega))}\leq C, \qquad Re(\lambda)\geq\omega_0 \tag{2.8}$$

where C is independent of m, m_0, ε and λ.

Proof. Note that using Lemma 2.1, since $p>N$ and $r>N-1$ we can take $s+\sigma<2$ in (2.5) and (2.6) and then P_ε, Q_0, R_0 are well defined from $H^{s,q}(\Omega)$ into $H^{-\sigma,q}(\Omega)$ provided $s>1/q$, $\sigma>1/q'$ and

$$2>s+\sigma>\max\{\frac{N}{p},1+\frac{N-1}{r}\}:=K.$$

In particular, for any $\sigma \in (\frac{1}{q'}, 2 - \frac{1}{q})$ there exists $\tilde{\sigma} < \sigma$ such that

$$S_\varepsilon := P_\varepsilon + Q_0 + R_0 : H^{2-\sigma,q}(\Omega) \to H^{-\tilde{\sigma},q}(\Omega) \subset H^{-\sigma,q}(\Omega) \tag{2.9}$$

is continuous and uniformly bounded in norm, for m and m_0 in bounded sets and $0 < \varepsilon \le \varepsilon_0$.

Then for given $g \in H^{-\sigma,q}_{bc}(\Omega)$ the equation $A_0 u + \lambda u - S_\varepsilon u = g$ can be written as

$$u = T^\varepsilon_\lambda(u) := (A_0 + \lambda I)^{-1} g + (A_0 + \lambda I)^{-1} S_\varepsilon u.$$

Observe now that from the resolvent estimates in [3], Chapter I, Section 1.2, we have that for each $0 \le \alpha \le 1$, for some $\omega > 0$ and $C \ge 1$,

$$\|(A_0 + \lambda)^{-1}\|_{\mathcal{L}(H^{-\alpha,q}_{bc}(\Omega), H^{-\alpha,q}_{bc}(\Omega))} \le \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda) \ge \omega$$

and

$$\|(A_0 + \lambda)^{-1}\|_{\mathcal{L}(H^{-\alpha,q}_{bc}(\Omega), H^{2-\alpha,q}_{bc}(\Omega))} \le C, \qquad \mathrm{Re}(\lambda) \ge \omega.$$

Interpolating these inequalities we get, for any $\tilde{\alpha} > \alpha$

$$\|(A_0 + \lambda)^{-1}\|_{\mathcal{L}(H^{-\alpha,q}(\Omega), H^{2-\tilde{\alpha},q}(\Omega))} \le \frac{C}{|\lambda|^{(\tilde{\alpha}-\alpha)/2}}.$$

Therefore, from this and (2.9) we get that the Lipschitz constant of $T^\varepsilon_\lambda : H^{2-\sigma,q}_{bc}(\Omega) \to H^{2-\sigma,q}_{bc}(\Omega)$ is bounded by $\frac{C}{|\lambda|^{(\sigma-\tilde{\sigma})/2}}$.

Therefore there exists $\omega_0 \ge \omega$ such that T^ε_λ is a contraction, with Lipschitz constant $\theta < 1$ uniform for all $\mathrm{Re}(\lambda) \ge \omega_0$ and m, m_0 as in the statement and $0 < \varepsilon \le \varepsilon_0$. This implies that the unique fixed point of T^ε_λ satisfies

$$\|u\|_{H^{2-\sigma,q}_{bc}(\Omega)} \le \frac{1}{1-\theta} \|(A_0 + \lambda)^{-1} g\|_{H^{2-\sigma,q}_{bc}(\Omega)} \le \frac{C}{1-\theta} \|g\|_{H^{-\sigma,q}_{bc}(\Omega)}, \tag{2.10}$$

which proves (2.8). This, in turn, implies

$$\|u\|_{H^{-\sigma,q}_{bc}(\Omega)} \le \|(A_0+\lambda)^{-1} g\|_{H^{-\sigma,q}_{bc}(\Omega)} + \|(A_0+\lambda)^{-1} S_\varepsilon u\|_{H^{-\sigma,q}_{bc}(\Omega)} \le \frac{C}{|\lambda|} (\|g\|_{H^{-\sigma,q}_{bc}(\Omega)} + \|S_\varepsilon u\|_{H^{-\sigma,q}_{bc}(\Omega)})$$

and, using again (2.9) and (2.10), we get

$$\|u\|_{H^{-\sigma,q}_{bc}(\Omega)} \le \frac{C}{|\lambda|} \|g\|_{H^{-\sigma,q}_{bc}(\Omega)}$$

which proves (2.7). ■

Remark 2.3 *Note that if we only consider interior potentials, that is if $m_0 = 0$ and $V_\varepsilon = 0$, then the range of σ in Theorem 2.2 changes to $\sigma \in (0, 2)$, since we do not have the restriction (2.6).*

We have then the following corollaries

Corollary 2.4
i) Assume

$$m_\varepsilon \to m \quad \text{in } L^p(\Omega), \quad p > \frac{N}{2},$$
$$m_{0,\varepsilon} \to m_0 \quad \text{in } L^r(\Gamma), \quad r > N-1,$$
$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \text{for some } r > N-1.$$

Assume moreover that

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon \to h_0 \quad cc - L^q \quad \text{for some } q > 1.$$

and

$$g_\varepsilon \to g_0 \text{ weakly in } L^z(\Omega), \quad j_\varepsilon \to j_0 \text{ weakly in } L^t(\Gamma)$$

for some $z \geq Nq/(N-1+q)$ and $t \geq q$.

Then, there exists some $\omega_0 > 0$ independent of ε, such that for $Re(\lambda) \geq \omega_0$ there exists a unique solution, u^ε, of

$$\left\{\begin{array}{rcll} -div(a(x)\nabla u^\varepsilon) + c(x)u^\varepsilon &=& m_\varepsilon(x)u^\varepsilon + \lambda u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} h_\varepsilon + g_\varepsilon & \text{in } \Omega, \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon &=& m_{0,\varepsilon}(x)u^\varepsilon + j_\varepsilon & \text{on } \Gamma, \\ \mathcal{B}u^\varepsilon &=& 0 & \text{on } \partial\Omega \setminus \Gamma, \end{array}\right.$$

which converges

$$u^\varepsilon \to u \quad \text{in} \quad H^{s,q}(\Omega)$$

for any $s < 1 + \frac{1}{q}$ where u is the unique solution of the limiting problem

$$\left\{\begin{array}{rcll} -div(a(x)\nabla u) + c(x)u &=& m(x)u + \lambda u + g_0 & \text{in } \Omega, \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& (m_0(x) + V_0(x))u + h_0 + j_0 & \text{on } \Gamma, \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \setminus \Gamma. \end{array}\right.$$

In particular, if $q > N-1$, $z > N/2$ and $t > N-1$, then

$$u^\varepsilon \to u \quad \text{in} \quad C^\beta(\overline{\Omega}),$$

for some $\beta > 0$.
ii) If $m \in L^p(\Omega)$, with $p > \frac{N}{2}$ and $m_0 \in L^r(\Gamma)$ with $r > N-1$ then for any $1 < q < \infty$, the operator $A_0 - (Q_0 + R_0)$ in Theorem 2.2 is resolvent positive. That is, there exists some $\omega_0 > 0$, such that for any $\lambda \geq \omega_0$ and $\sigma \in (\frac{1}{q'}, 2 - \frac{1}{q})$,

$$\text{if} \quad 0 \leq g \in H^{-\sigma,q}_{bc}(\Omega) \quad \text{then} \quad 0 \leq (A_0 + \lambda I - (Q_0 + R_0))^{-1} g \in H^{2-\sigma,q}_{bc}(\Omega).$$

The constant ω_0 can be taken uniform for m lying in a bounded set in $L^p(\Omega)$, with $p > N/2$ and m_0 lying in a bounded set in $L^r(\Gamma)$, with $r > N-1$.

Proof.
i) Note that since $V_0 \in L^r(\Gamma)$, we define

$$< P_0 u, \varphi >= \int_\Gamma V_0 u \varphi \tag{2.11}$$

for suitable u and φ. Then it was proved in Lemma 2.5 in [5] that

$$P_\varepsilon \to P_0 \quad \text{in} \quad \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))$$

for s, σ as in (2.6). The rest of the proof goes along the same lines as Corollary 3.2 in [5].
ii) Note that for C^1 coefficients, the property of resolvent positive follows from Theorem 8.7, page 48, in [2]. For nonsmooth coefficients take sequences of C^1 smooth functions in Ω and Γ respectively such that $m_\varepsilon \to m$ in $L^p(\Omega)$ with $p > \frac{N}{2}$, and $m_{0,\varepsilon} \to m_0$ in $L^r(\Gamma)$, with $r > N-1$ and apply part i) with $V_\varepsilon = 0$, $h_\varepsilon = 0$, $j_\varepsilon = 0$, $g_\varepsilon = g$. ■

Remark 2.5
i) Note that part i) of Corollary 2.4 can be stated as

$$(A_0 + \lambda I - (P_\varepsilon + Q_0 + R_0))^{-1} \to (A_0 + \lambda I - (P_0 + Q_0 + R_0))^{-1}$$

in $\mathcal{L}(H^{-\sigma,q}_{bc}(\Omega), H^{2-\sigma,q}_{bc}(\Omega))$.
ii) On the other hand the convergence of the resolvent above implies also that the spectrum of the operators $A_0 - (P_\varepsilon + Q_0 + R_0)$ *and* $A_0 - (P_0 + Q_0 + R_0)$ *are close. See Corollary 4.2 and Remark 4.3 in [5] or [8] for a precise statement.*

Corollary 2.6 *Assume that m lies in a bounded set in $L^p(\Omega)$, with $p > N/2$, m_0 lies in a bounded set in $L^r(\Gamma)$ and also that the family of potentials V_ε is a L^r–concentrated bounded family, for $r > N-1$, that is*

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |V_\varepsilon|^r \le C, \quad r > N-1.$$

Then, for any $1 < q < \infty$, and for $\sigma \in (\frac{1}{q'}, 2 - \frac{1}{q})$, $0 < \varepsilon \le \varepsilon_0$, the operator $-(A_0 - (P_\varepsilon + Q_0 + R_0))$ in $H^{-\sigma,q}_{bc}(\Omega)$ with domain $H^{2-\sigma,q}_{bc}(\Omega)$, generates an strongly continuous, order preserving, analytic semigroup, $S_{m,m_0,\varepsilon}(t)$. Moreover the semigroup satisfies the smoothing estimates

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{H^{2\alpha,q}_{bc}(\Omega)} \le \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\beta,q}_{bc}(\Omega) \tag{2.12}$$

for $-\frac{1}{2q'} \ge \beta \ge -1 + \frac{1}{2q}$ and $1 - \frac{1}{2q'} \ge \alpha \ge \beta$ for some $M_{\alpha,\beta}$ and $\mu \in I\!R$ independent of m, m_0 and $0 < \varepsilon \le \varepsilon_0$. In particular, one has

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{L^\tau(\Omega)} \le \frac{M_{\rho,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \qquad t > 0, \quad u_0 \in L^\rho(\Omega) \tag{2.13}$$

for $1 < \rho \leq \tau \leq \infty$ with $M_{\rho,\tau}$ and μ independent of m, m_0 and $0 < \varepsilon \leq \varepsilon_0$.

Finally, for every $u_0 \in H^{2\beta,q}_{bc}(\Omega)$, with β as above, the function $u(t; u_0) := S_{m,m_0,\varepsilon}(t)u_0$ is, for $t > 0$, a weak solution of (1.3) in the sense that

$$\int_\Omega u_t\varphi + \int_\Omega (a(x)\nabla u\nabla\varphi + (c(x) - m(x))u)\varphi + \int_\Gamma (b(x) - m_0(x))u\varphi = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} V_\varepsilon(x)u\varphi \tag{2.14}$$

for all sufficiently smooth φ.

Proof. Using the uniform resolvent estimates, (2.7) and (2.8) in Theorem 2.2, we use Proposition 1.4.1, Chapter I, in [3] to obtain similar uniform estimates for λ in a uniform sector in the complex plane.

Then using that the semigroup can be obtained as an integral of the resolvent over a suitable contour around that sector in the complex plane, see e.g. Theorem 1.3.4 in [6], we obtain an analytic semigroup $S_{m,m_0,\varepsilon}(t)$ that satisfies

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{H^{\alpha,q}_{bc}(\Omega)} \leq \frac{M_{\alpha,\sigma}e^{\mu t}}{t^{\frac{\alpha+\sigma}{2}}}\|u_0\|_{H^{-\sigma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{-\sigma,q}_{bc}(\Omega)$$

for $2 - \sigma \geq \alpha \geq -\sigma$ and for some $M_{\alpha,\sigma}$ and μ independent of m, m_0 and $0 < \varepsilon \leq \varepsilon_0$.

As σ ranges in $(\frac{1}{q'}, 2 - \frac{1}{q})$ and some easy reiteration of the estimates above, we get (2.12).

Now, (2.12) and the sharp Sobolev embeddings of $H^{2\beta,q}_{bc}(\Omega) \supset L^\rho(\Omega)$ (since β is negative) and $H^{2\alpha,q}_{bc}(\Omega) \subset L^\tau(\Omega)$, with $\alpha \geq 0$, give

$$-1 - \frac{N-1}{q} \geq \frac{-N}{\rho} := 2\beta - \frac{N}{q} \geq -2 - \frac{N-1}{q}$$

and

$$1 - \frac{N-1}{q} \geq \frac{-N}{\tau} := 2\alpha - \frac{N}{q} \geq -\frac{N}{q},$$

which gives (2.13) for $\rho, \tau \geq 1$ such that

$$\frac{Nq}{N+2q-1} \leq \rho \leq \frac{Nq}{N+q-1}$$

and

$$\rho \leq \tau \begin{cases} \leq \frac{Nq}{N-q-1} & \text{if } q < N-1 \\ < \infty, & \text{if } q = N-1 \\ \leq \infty & \text{if } q > N-1. \end{cases}$$

Using q as a parameter, some reiteration of the argument above gives (2.13) for $1 < \rho \leq \tau \leq \infty$.

The order preserving property of the semigroups follows from the positivity of the resolvent established in Corollary 2.4.

Finally, since the semigroup is analytic, for every $u_0 \in H^{2\beta,q}_{bc}(\Omega)$, with β as in (2.12), the function $u(t; u_0) := S_{m,m_0,\varepsilon}(t)u_0$ satisfies, for $t > 0$, $u_t + (A_0 - (P_\varepsilon + Q_0 + R_0))u = 0$ in $H^{2\beta,q}_{bc}(\Omega)$. Then, (2.3), (2.4) and the characterization of the realization of A_0 in that spaces given in Section 8 in [2], gives (2.14). ∎

Remark 2.7 *Note that if $V_0 \in L^r(\Gamma)$, for $r > N-1$, with the choice $V_\varepsilon = 0$ and $m_0 + V_0$ replacing m_0, Corollary 2.6 allows to define the semigroup $S_{m,m_0+V_0}(t)$ such that for every $u_0 \in H^{2\beta,q}_{bc}(\Omega)$, with β as in the Corollary, the function $u(t;u_0) := S_{m,m_0+V_0}(t)u_0$ is, for $t > 0$, a weak solution of (1.4) in the sense that*

$$\int_\Omega u_t\varphi + \int_\Omega (a(x)\nabla u\nabla\varphi + (c(x) - m(x))u\varphi + \int_\Gamma (b(x) - m_0(x))u = \int_\Gamma V_0(x)u\varphi \quad (2.15)$$

for all sufficiently smooth φ.

3 Perturbation of linear analytic semigroups in scales of Banach spaces

Observe that in Corollary 2.6 the space of initial data, $H^{2\beta,q}_{bc}(\Omega)$, has always negative exponent since $-\frac{1}{2q'} \geq \beta \geq -1 + \frac{1}{2q}$. Therefore, our goal in this section is to improve Corollary 2.6 by enlarging the range for which (2.12) is satisfied to

$$\alpha, \beta \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'}), \quad \alpha \geq \beta,$$

with constants independent of m, m_0 and $0 < \varepsilon \leq \varepsilon_0$.

Instead of relying on resolvent estimates, as in the previous section (see Theorem 2.2) we use a "parabolic" approach to deal with the perturbations. This approach will also be useful to obtain the semigroup $S_{m,m_0,\varepsilon}(t)$ as a perturbation of $S_0(t)$ and to analyze the convergence as $\varepsilon \to 0$. Also, this approach can be applied to many other perturbation problems.

For this, note that (2.1), that is

$$\|S_0(t)u_0\|_{H^{2\alpha,q}_{bc}(\Omega)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\beta,q}_{bc}(\Omega) \quad (3.1)$$

for $1 \geq \alpha \geq \beta \geq -1$, can be rewritten in an abstract language as follows. For this we will consider below

$$X^\alpha := H^{2\alpha,q}_{bc}(\Omega), \quad \alpha \in I := [-1, 1]. \quad (3.2)$$

In view of (3.1) and (3.2), we consider a linear analytic semigroup $S(t)$ defined on each of the spaces of the family of Banach spaces (the "scale") $\{X^\alpha\}_{\alpha\in I}$ where I is an interval of real indexes. The norm of the space X^α is denoted by $\|\cdot\|_\alpha$.

We also assume that for all $\alpha, \beta \in I$ with $\alpha \geq \beta$ we have

$$X^\alpha \subset X^\beta \quad (3.3)$$

with continuous inclusion and the norm of the inclusion will be denoted $\|i\|_{\alpha,\beta}$. Note that for the example above, (3.2), we have $\|i\|_{\alpha,\beta} \leq 1$ for all α, β.

We also assume the semigroup acting on the scale satisfies, $\alpha, \beta \in I$ with $\alpha \geq \beta$

$$\|S(t)\|_{\beta,\alpha} := \|S(t)\|_{\mathcal{L}(X^\beta, X^\alpha)} \leq \frac{M_0(\beta,\alpha)}{t^{\alpha-\beta}}, \qquad \text{for all} \quad 0 < t \leq 1 \tag{3.4}$$

for some constant $M_0(\beta,\alpha) > 0$.

Remark 3.1
i) Note that the semigroup $S_0(t)$ of Section 2 in the scale (3.2) satisfies that for each β, the domain of the generator $-A_0$ in X^β is given by $D(A_0) = X^{\beta+1}$ and also the inclusion (3.3) is dense and compact. These properties will not be used below.
ii) In view of (2.2) another possible scale for the semigroup $S_0(t)$ is the scale of Lebesgue spaces. More precisely we can set

$$X^\alpha = L^\rho(\Omega), \quad 1 \leq q < \infty, \quad \alpha = -\frac{N}{2q} \in I := [-N/2, 0).$$

On this scale we have (3.3) (which is dense but not compact) and $S_0(t)$ satisfies (3.4) but not the property in part i) of this Remark.

Observe that from these assumptions we get

Lemma 3.2 *Assume (3.3) and (3.4) are satisfied. Then*
i) For every $\alpha, \beta \in I$ and $\alpha \geq \beta$ and for all $T > 0$,

$$\|S(t)\|_{\beta,\alpha} \leq \frac{M_0(\beta,\alpha,T)}{t^{\alpha-\beta}}, \qquad \textit{for all} \quad 0 < t \leq T \tag{3.5}$$

for some constant $M_0(\beta,\alpha,T) > 0$.
ii) For each $\beta \in I$ there exists $\omega(\beta) \geq 0$ such that

$$\|S(t)\|_{\beta,\beta} \leq M_0(\beta,\beta)\mathrm{e}^{\omega(\beta)t}, \quad \textit{for all} \quad t > 0.$$

iii) Moreover, if for some fixed $\beta_0 \in I$, we have

$$\|S(t)\|_{\beta_0,\beta_0} \leq M\mathrm{e}^{\omega_0 t}, \quad \textit{for all} \quad t > 0 \tag{3.6}$$

for some $M = M(\beta_0)$ and $\omega_0 \in I\!R$, then for any $\alpha \in I$, there exists a constant $M(\alpha) \geq 1$ such that

$$\|S(t)\|_{\alpha,\alpha} \leq M(\alpha)\mathrm{e}^{\omega_0 t}, \quad \textit{for all} \quad t > 0. \tag{3.7}$$

Moreover, given $t_0 > 0$, define $\delta = \|S(t_0)\|_{\beta_0,\beta_0}$. Then we have (3.6) with

$$\omega_0 = \frac{\ln(\delta)}{t_0}$$

and some constant M depending on t_0, δ and $M_0(\beta_0,\beta_0,t_0)$ as in (3.5). In particular if $\delta < 1$ then $\omega_0 < 0$.

iv) In the situation of iii), for every $\alpha, \beta \in I$ and $\alpha \geq \beta$ we have

$$\|S(t)\|_{\beta,\alpha} \leq \begin{cases} M_1(\beta,\alpha)t^{-(\alpha-\beta)} & \text{if } 0 < t \leq 1, \\ M_1(\beta,\alpha)\mathrm{e}^{\omega_0 t} & \text{if } t > 1 \end{cases}$$

for some positive constant $M_1(\beta, \alpha)$.

In particular, for all $\varepsilon > 0$ there exists $M_\varepsilon(\beta, \alpha) > 0$ such that

$$\|S(t)\|_{\beta,\alpha} \leq M_\varepsilon(\beta,\alpha)\frac{\mathrm{e}^{(\omega_0+\varepsilon)t}}{t^{\alpha-\beta}}, \quad \textit{for all} \quad t > 0.$$

Proof.

i) Indeed, given $T > 0$ define n as the smallest integer such that $T \leq n + 1$. Then, for $0 < t \leq T$, define $h = \frac{t}{n+1} \leq 1$ and $s_j = jh$, $j = 0, \ldots, n + 1$. Thus $s_{n+1} = t$ and, since

$$S(t) = S(s_{n+1} - s_n) \cdots S(s_1 - s)$$

we get, form (3.4),

$$\|S(t)\|_{\beta,\alpha} \leq M_0(\alpha,\alpha)^n M_0(\beta,\alpha)(n+1)^{\alpha-\beta}t^{-(\alpha-\beta)} \qquad \text{for all} \quad 0 < t \leq T.$$

Hence we can take

$$M_0(\beta,\alpha,T) = M_0(\alpha,\alpha)^n M_0(\beta,\alpha)(n+1)^{\alpha-\beta}.$$

ii) In particular, with $\alpha = \beta$, given $t > 0$ define $n \in I\!N$ such that $n \leq t < n + 1$ and we get as above,

$$\|S(t)\|_{\beta,\beta} \leq M_0(\beta,\beta)^{n+1} \leq M_0(\beta,\beta)^{t+1} \leq M_0(\beta,\beta)\mathrm{e}^{\ln(M_0(\beta,\beta))t}, \quad \text{for all} \quad t > 0$$

Note that as $M_0(\beta, \beta) \geq 1$ then $\omega(\beta) := \ln(M_0(\beta, \beta)) \geq 0$.

iii) First notice that from (3.4), for any $\alpha \geq \beta_0$, we have $\|S(1)\|_{\beta_0,\alpha} \leq M_0(\beta_0, \alpha)$. Now, if $t > 1$, then

$$\begin{aligned} \|S(t)u_0\|_\alpha &\leq \|S(1)\|_{\beta_0,\alpha}\|S(t-1)u_0\|_{\beta_0} \\ &\leq M_0(\beta_0,\alpha)M\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t}\|u_0\|_{\beta_0} \\ &\leq M_0(\beta_0,\alpha)\|i\|_{\alpha,\beta_0}M\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t}\|u_0\|_\alpha, \end{aligned}$$

where $\|i\|_{\alpha,\beta_0}$ denotes the norm of the inclusion $X^\alpha \hookrightarrow X^{\beta_0}$. Thus,

$$\|S(t)\|_{\alpha,\alpha} \leq K\mathrm{e}^{\omega_0 t}, \qquad \text{for all} \quad t > 1$$

with $K = M_0(\beta_0, \alpha)\|i\|_{\alpha,\beta_0}M\mathrm{e}^{-\omega_0}$.

On the other hand, if $\beta_0 \geq \alpha$, we also have, from (3.4), $\|S(1)\|_{\alpha,\beta_0} \leq M_0(\alpha, \beta_0)$ and for $t > 1$,

$$\begin{aligned} \|S(t)u_0\|_\alpha &\leq \|i\|_{\beta_0,\alpha}\|S(t)u_0\|_{\beta_0} \\ &\leq \|i\|_{\beta_0,\alpha}\|S(t-1)\|_{\beta_0,\beta_0}\|S(1)u_0\|_{\beta_0} \\ &\leq \|i\|_{\beta_0,\alpha}M\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t}\|S(1)\|_{\alpha,\beta_0}\|u_0\|_\alpha \\ &\leq \|i\|_{\beta_0,\alpha}M\mathrm{e}^{-\omega_0}M_0(\alpha,\beta_0)\mathrm{e}^{\omega_0 t}\|u_0\|_\alpha. \end{aligned}$$

Thus,

$$\|S(t)\|_{\alpha,\alpha} \leq K\mathrm{e}^{\omega_0 t}, \qquad \text{for all} \quad t > 1$$

with $K = M_0(\alpha, \beta_0)\|i\|_{\beta_0,\alpha} M\mathrm{e}^{-\omega_0}$.

Therefore, for any $\alpha \in I$, we have the estimate

$$\|S(t)\|_{\alpha,\alpha} \leq K(\alpha)\mathrm{e}^{\omega_0 t}, \qquad \text{for all} \quad t > 1.$$

Hence, again from (3.4) with $\beta = \alpha$, we get (3.7) with

$$M(\alpha) = \begin{cases} \max\{K(\alpha), M_0(\alpha,\alpha)\} & \text{if } \omega_0 \geq 0 \\ \max\{K(\alpha), M_0(\alpha,\alpha)\mathrm{e}^{-\omega_0}\} & \text{if } \omega_0 \leq 0. \end{cases}$$

If moreover for given $t_0 > 0$ we define $\delta = \|S(t_0)\|_{\beta_0,\beta_0}$ then for $t > 0$ we write $t = nt_0 + s$, with $n \in I\!N$ and $0 \leq s < t_0$. Then

$$\|S(t)\|_{\beta_0,\beta_0} \leq \delta^n \|S(s)\|_{\beta_0,\beta_0} \leq \mathrm{e}^{\ln(\delta)(\frac{t-s}{t_0})} M_0(\beta_0, \beta_0, t_0)$$

with $M_0(\beta_0, \beta_0, t_0)$ as in (3.5) and the result follows. In particular if $\delta < 1$ then $\omega_0 < 0$.
iv) Now note that if $0 < t \leq 1$, the estimate reduces to (3.4). On the other hand, if $t > 1$, then, using (3.4) and part iii), we get

$$\begin{aligned} \|S(t)\|_{\beta,\alpha} &\leq \|S(t-1)\|_{\alpha,\alpha}\|S(1)\|_{\beta,\alpha} \\ &\leq M_0(\beta,\alpha)M(\alpha)\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t} = M_1(\beta,\alpha)\mathrm{e}^{\omega_0 t}. \end{aligned}$$

and the rest follows easily. ■

Remark 3.3 *Observe that if the original constants $M_0(\beta, \alpha)$ in (3.4), do not depend (or can be taken independent of $\alpha, \beta \in I$), then the same is true for $M_0(\beta, \alpha, T)$ and $M(\alpha)$ in (3.7) depends on the scale only through the norm of the inclusions $\|i\|_{\beta_0,\alpha}$ or $\|i\|_{\alpha,\beta_0}$.*

Hereafter we will make use extensively the following spaces.

Definition 3.4

For $T > 0$, $\gamma \in I$ and $\varepsilon \geq 0$ we define for functions in $L^\infty_{loc}((0,T], X^\gamma)$, the quantity

$$|||u|||_{\gamma,\varepsilon} = \sup_{t\in(0,T]} t^\varepsilon \|u(t)\|_\gamma$$

which becomes a norm on the set of functions where it is finite, that we denote $\mathcal{L}^\infty_\varepsilon((0,T], X^\gamma)$.

Note that this set always contains $L^\infty([0,T], X^\gamma)$ and coincides with it when $\varepsilon = 0$. Also, the spaces are increasing with ε. Then we have

Lemma 3.5 *For $T > 0$, $\gamma \in I$ and $\varepsilon \geq 0$, $\mathcal{L}^\infty_\varepsilon((0,T], X^\gamma)$ with norm $|||u|||_{\gamma,\varepsilon}$ is a Banach space.*

Proof. Note that $\{u^k\}_k$ is a Cauchy sequence in $\mathcal{L}^\infty_\varepsilon((0,T],X^\gamma)$ iff $v^k(t) = t^\varepsilon u^k(t)$ is a Cauchy sequence in $L^\infty([0,T],X^\gamma)$ and also $u^k(t)$ converges in X^γ to some $u(t)$ for each $t > 0$ and uniformly for $\delta \leq t \leq T$. The rest is easy. ■

Also, part i) in Lemma 3.2 can be stated as

Lemma 3.6 *Assume the semigroup $S(t)$ and the scale of spaces satisfy (3.3) and (3.4). Then, for any $\alpha, \beta \in I$ with $\alpha \geq \beta$ and $T > 0$,*

$$S(\cdot) : X^\beta \longrightarrow \mathcal{L}^\infty_{\alpha-\beta}((0,T],X^\alpha), \qquad u_0 \mapsto S(\cdot)u_0$$

is linear and continuous.

Remark 3.7 *To motivate our approach below, observe that from the proof of Corollary 2.6 we have that for every $u_0 \in H^{2\beta,q}_{bc}(\Omega)$, with β as in (2.12), the function $u(t;u_0) := S_{m,m_0,\varepsilon}(t)u_0$ satisfies, for $t > 0$, $u_t + A_0 u = S_\varepsilon u$ in $H^{2\beta,q}_{bc}(\Omega)$, with $S_\varepsilon = P_\varepsilon + Q_0 + R_0$ as in (2.9). Then, the variations of constants formula for the analytic semigroup $S_0(t)$, see [6], gives*

$$u(t;u_0) = S_0(t)u_0 + \int_0^t S_0(t-\tau)S_\varepsilon u(\tau;u_0)\,d\tau.$$

Now, in the general setting (3.3)–(3.4), assume that for some fixed $\alpha \geq \beta$, with $0 \leq \alpha - \beta < 1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X^\alpha, X^\beta). \tag{3.8}$$

Consider the abstract linear integral problem with u_0 to be chosen below

$$u(t;u_0) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau;u_0)\,d\tau, \qquad t > 0. \tag{3.9}$$

Definition 3.8 *For a given function u defined on $(0,T]$ and taking values in X^α, we define*

$$\mathcal{F}(u,u_0)(t) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau)\,d\tau, \qquad 0 < t \leq T \tag{3.10}$$

assumed it is well defined.

Then we have the following Lemma

Lemma 3.9 *Assume the semigroup $S(t)$ and the scale of spaces satisfy (3.3) and (3.4) and the perturbation P satisfies (3.8). Assume $\varepsilon \geq 0$, $\delta \geq 0$, $\gamma, \gamma' \in I$, with and $\gamma' \geq \gamma$, are such that*

$$\beta \leq \gamma' < \beta + 1 \quad \text{and} \quad 0 \leq \varepsilon < 1 \tag{3.11}$$

Then for $u \in \mathcal{L}^\infty_\varepsilon((0,T],X^\alpha)$ and $u_0 \in X^\gamma$, we have
i) For $0 < t \leq T$

$$t^\delta \| \int_0^t S(t-\tau)Pu(\tau)\,d\tau \|_{\gamma'} \leq M_1(T) t^{\beta+\delta+1-\gamma'-\varepsilon} \|P\|_{\mathcal{L}(X^\alpha,X^\beta)} |||u|||_{\alpha,\varepsilon}$$

where $M_1(T) = c(\beta, \gamma', \varepsilon) M_0(\beta, \gamma', T)$.
ii) For $0 < t \leq T$

$$t^{\delta} \|\mathcal{F}(u, u_0)(t)\|_{\gamma'} \leq t^{\delta} \|S(t)u_0\|_{\gamma'} + M_1(T) t^{\beta+\delta+1-\gamma'-\varepsilon} \|P\|_{\mathcal{L}(X^{\alpha}, X^{\beta})} |||u|||_{\alpha,\varepsilon}$$

with $M_1(T)$ *as above.*
iii) In particular, if

$$\delta = \gamma' - \gamma \geq 0 \quad \textit{and} \quad \gamma < \beta + 1 - \varepsilon, \tag{3.12}$$

then

$$|||\mathcal{F}(u, u_0)|||_{\gamma',\delta} \leq |||S(\cdot)u_0|||_{\gamma',\delta} + C(T) \|P\|_{\mathcal{L}(X^{\alpha}, X^{\beta})} |||u|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T) T^{\beta+1-\gamma-\varepsilon}$ *and all terms above are finite. In particular,*

$$(u, u_0) \ni \mathcal{L}^{\infty}_{\varepsilon}((0, T], X^{\alpha}) \times X^{\gamma} \longmapsto \mathcal{F}(u, u_0) \in \mathcal{L}^{\infty}_{\gamma'-\gamma}((0, T], X^{\gamma'})$$

is linear and continuous.

Proof. We first prove part i), and then part ii) and iii) are immediate. Using (3.5) we have for $\gamma' \geq \beta$

$$t^{\delta} \| \int_0^t S(t-\tau) P u(\tau)\, d\tau \|_{\gamma'} \leq M(T) t^{\delta} \int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta}} \|P\|_{\alpha,\beta} \|u(\tau)\|_{\alpha}\, d\tau \leq$$

$$\leq M(T) |||u|||_{\alpha,\varepsilon} \|P\|_{\alpha,\beta} t^{\delta} \int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta} \tau^{\varepsilon}}\, d\tau,$$

where we have set $M(T) = M_0(\beta, \gamma', T)$ as in (3.5). Now the change of variables $\tau = rt$ gives the result with

$$M_1(T) = M(T) \left(\int_0^1 \frac{1}{(1-r)^{\gamma'-\beta} r^{\varepsilon}}\, dr \right)$$

provided $\gamma' - \beta < 1$ and $\varepsilon < 1$ as in the statement. ■

Note that when we take $\gamma' > \gamma$ in Lemma 3.9 above, this result can be interpreted as a smoothing effect of the variation of constants formula (3.10). The same applies to the next result in which we analyze continuity in time.

Lemma 3.10 *With the same notations and assumptions as in Lemma 3.9, for* $u \in \mathcal{L}^{\infty}_{\varepsilon}((0, T], X^{\alpha})$ *and* $u_0 \in X^{\gamma}$*, if (3.11) holds, that is*

$$\beta \leq \gamma' < \beta + 1, \quad 0 \leq \varepsilon < 1$$

we have

$$\mathcal{F}(u, u_0) \in C((0, T], X^{\gamma'}).$$

Further more $\mathcal{F}(u, u_0)$ *is locally Hölder continuous with values in* $X^{\gamma'}$.

Proof. Fix $0 < t < T$ and take $h > 0$ small, so that $t + h \le T$. Also take $0 < t^* < t - h$ to be chosen below. Then, from (3.10) we have

$$\mathcal{F}(u, u_0)(t^*) = S(t^*)u_0 + \int_0^{t^*} S(t^* - \tau)Pu(\tau)\, d\tau.$$

Then we get,

$$\mathcal{F}(u, u_0)(t + h) = S(t + h - t^*)\mathcal{F}(u, u_0)(t^*) + \int_{t^*}^{t+h} S(t + h - \tau)Pu(\tau)\, \mathrm{d}\tau,$$

$$\mathcal{F}(u, u_0)(t) = S(t - t^*)\mathcal{F}(u, u_0)(t^*) + \int_{t^*}^{t} S(t - \tau)Pu(\tau)\, \mathrm{d}\tau$$

The, suppressing temporarily the dependence in u_0, we get

$$\mathcal{F}(u)(t + h) - \mathcal{F}(u)(t) = \Big(S(t + h - t^*) - S(t - t^*)\Big)\mathcal{F}(u)(t^*)+$$

$$+\int_t^{t+h} S(t + h - \tau)Pu(\tau)\, d\tau + \int_{t^*}^{t} \Big(S(h) - I\Big)S(t - \tau)Pu(\tau)\, d\tau. \tag{3.13}$$

Now we estimate in norm in (3.13) to get

$$\|\mathcal{F}(u)(t + h) - \mathcal{F}(u)(t)\|_{\gamma'} \le \|\Big(S(t + h - t^*) - S(t - t^*)\Big)\mathcal{F}(u)(t^*)\|_{\gamma'}+$$

$$+M(T) \int_t^{t+h} (t + h - \tau)^{-(\gamma'-\beta)}\|P\|_{\alpha,\beta}\|u(\tau)\|_{\alpha}\, d\tau + M(T) \int_{t^*}^{t} (t - \tau)^{-(\gamma'-\beta)}\|P\|_{\alpha,\beta}\|u(\tau)\|_{\alpha}\, d\tau$$

where, in the third term, we have used that $\|S(h) - I\|_{\gamma',\gamma'}$ is bounded.

Now, since $S(t)$ is an analytic semigroup, the first term is bounded by a constant times h, while using that u is bounded in X^{α} on $[t^*, T]$, the second and third ones are bounded, respectively, by

$$K(T, u) \left(\int_t^{t+h} (t + h - \tau)^{-(\gamma'-\beta)}\, d\tau\right) \|P\|_{\mathcal{L}(X^{\alpha},X^{\beta})} = K_1(T, u)\|P\|_{\mathcal{L}(X^{\alpha},X^{\beta})}h^{1-(\gamma'-\beta)}$$

$$K(T, u) \left(\int_{t^*}^{t} (t - \tau)^{-(\gamma'-\beta)}\, d\tau\right) \|P\|_{\mathcal{L}(X^{\alpha},X^{\beta})} = K_1(T, u)\|P\|_{\mathcal{L}(X^{\alpha},X^{\beta})}(t - t^*)^{1-(\gamma'-\beta)}$$

Now taking $t^* = t - 2h$, we get the result. ■

Now we finally analyze continuity at $t = 0$.

Lemma 3.11 *With the notations of Lemma 3.9, if*

$$\beta \le \gamma' < \beta + 1 - \varepsilon, \quad 0 \le \varepsilon < 1$$

then for $u \in \mathcal{L}^{\infty}_{\varepsilon}((0, T], X^{\alpha})$ and $u_0 \in X^{\gamma'}$

$$\mathcal{F}(u, u_0)(t) \to u_0, \quad in \quad X^{\gamma'}, \quad as \quad t \to 0.$$

Moreover, if $u_0 \in X^{\gamma}$, for some $\gamma \le \gamma'$,

$$\mathcal{F}(u, u_0)(t) \to u_0, \quad in \quad X^{\gamma}, \quad as \quad t \to 0.$$

Proof. By part i) in Lemma 3.9, with $\delta = 0$, we have

$$\| \int_0^t S(t-\tau)Pu(\tau)\, d\tau \|_{\gamma'} \leq M_1(T) t^{\beta+1-\gamma'-\varepsilon} \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u|||_{\alpha,\varepsilon}$$

where $M_1(T) = c(\beta, \gamma', \varepsilon) M_0(\beta, \gamma', T)$. Clearly the right hand side above goes to zero, as $t \to 0$.

On the other hand note that, by the choice of u_0 we have $S(t)u_0 \to u_0$ in $X^{\gamma'}$, or in X^γ since the spaces satisfy (3.3). ■

To find solutions of the linear problem (3.9), we start by the following "base" case.

Proposition 3.12 Solutions in X^α.

Assume the semigroup $S(t)$ and the scale of spaces satisfy (3.3) and (3.4) and assume also the perturbation satisfies (3.8). If

$$0 \leq \alpha - \beta < 1, \tag{3.14}$$

then for each $u_0 \in X^\alpha$ there exists a unique solution of (3.9), $u(\cdot; u_0) \in L^\infty_{loc}((0, \infty), X^\alpha)$, which is moreover in $C([0, \infty), X^\alpha)$.

Furthermore, for each $\alpha \leq \gamma' < \beta + 1$, we have that the solution satisfies

$$u(\cdot; u_0) \in C((0, \infty), X^{\gamma'}).$$

Even more, the unique solutions of (3.9) define a linear semigroup in X^α as

$$S_P(t)u_0 := u(t; u_0), \quad \textit{for all} \quad t > 0 \tag{3.15}$$

Proof. We show that there exists $T > 0$ such that $\mathcal{F}(\cdot, u_0)$ is a contraction in $L^\infty([0, T], X^\alpha)$. For this take $u_0 \in X^\alpha$ and u_1, u_2 in $L^\infty([0, T], X^\alpha)$ and note that, the right hand side of (3.10) is affine in u. Also from (3.14) we can use part iii) of Lemma 3.9 with $\gamma' = \gamma = \alpha$, $\delta = \varepsilon = 0$, to get $\mathcal{F}(u_i, u_0) \in L^\infty([0, T], X^\alpha)$ and also

$$|||\mathcal{F}(u_1, u_0) - \mathcal{F}(u_2, u_0)|||_{\alpha,0} \leq C(T) \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u_1 - u_2|||_{\alpha,0}$$

with $C(T) = M_1(T) T^{\beta+1-\alpha}$ and is a contraction for small enough T.

Since T can be taken independent of $u_0 \in X^\alpha$, it is easy to obtain that the solutions are defined for all $t \geq 0$. The continuity in time comes from Lemma 3.10 while the continuity at $t = 0$ in X^α comes from Lemma 3.11, with $\gamma' = \alpha$ and $\varepsilon = 0$.

Also, from (3.10) it follows that the operators defined in (3.15) are linear. Finally, the continuity of $S_P(t)$ in X^α will be proved in Proposition 3.15 below. ■

For weaker initial data we have the following result.

Theorem 3.13 Solutions in X^γ.

Assume the scale of spaces satisfy (3.3) and (3.4) and assume also the perturbation satisfies (3.8). If (3.14) is satisfied, that is

$$0 \leq \alpha - \beta < 1,$$

then for each

$$\alpha - 1 < \gamma \leq \alpha, \tag{3.16}$$

there exists T *such that for each* $u_0 \in X^\gamma$ *there exists a unique solution of (3.9)* $u \in \mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$*, with* $0 \leq \varepsilon = \alpha - \gamma < 1$.

Moreover the solution above is defined for all $t > 0$ *and for each*

$$\beta \leq \gamma' < \beta + 1, \quad \gamma' \geq \gamma, \tag{3.17}$$

we have that the solution satisfies

$$u(\cdot; u_0) \in C((0,\infty), X^{\gamma'}).$$

If, additionally $u_0 \in X^{\gamma'}$ *then*

$$u(\cdot; u_0) \in C([0,\infty), X^{\gamma'}).$$

Even more, the unique solutions of (3.9) define a linear semigroup in X^γ *as*

$$S_P(t)u_0 := u(t; u_0), \quad \textit{for all} \quad t > s. \tag{3.18}$$

Proof. Now we show that $\mathcal{F}(\cdot, u_0)$ is a contraction in $\mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ with $0 \leq \varepsilon = \alpha - \gamma < 1$. For this take $u_0 \in X^\gamma$ and u_1, u_2 in $\mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ and note that the right hand side of (3.10) is affine in u. Also, from (3.14) and (3.16) we can use part iii) of Lemma 3.9 with $\gamma' = \alpha$ and $0 \leq \varepsilon = \delta = \alpha - \gamma < 1$, to get $\mathcal{F}(u_i, u_0) \in \mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ and also

$$|||\mathcal{F}(u_1, u_0) - \mathcal{F}(u_2, u_0)|||_{\alpha,\varepsilon} \leq C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)}|||u_1 - u_2|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T)T^{\beta+1-\alpha}$ and is a contraction for small enough T.

Since T can be taken independent of $u_0 \in X^\gamma$, then it is easy to obtain that the solutions are defined for all $t \geq 0$.

The continuity in time comes from Lemma 3.10 while the continuity at $t = 0$ in $X^{\gamma'}$ comes from Lemma 3.11, with $\varepsilon = \alpha - \gamma$.

Now observe that in particular we have that for $t_0 > 0$ the solution satisfies $u(t_0) \in X^\alpha$ and $u \in L^\infty_{loc}([t_0,\infty), X^\alpha)$. Hence after time t_0, the solution coincides with the unique solution of Proposition 3.12.

In particular, from this, it is easily seen that the linear operators $S_P(t)$ define a linear semigroup.

As before, the continuity of $S_P(t)$ in X^γ will be proved in Proposition 3.15 below. ■

Remark 3.14 *i) Note that the result in Proposition 3.12 and in Theorem 3.13 holds, even if*

$$\alpha - \beta = 1$$

provided the norm $\|P\|_{\mathcal{L}(X^\alpha, X^\beta)}$ *is sufficiently small.*

ii) Note that the time T *for which* $\mathcal{F}$ *is a contraction in Proposition 3.12 and in Theorem 3.13 can be taken the same for all perturbations such that*

$$\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0$$

for some $R_0 > 0$.

Now we prove the following estimates on the solutions of (3.9). In particular this proves that the semigroup $S_P(t)$ defined in (3.15) and (3.18) is continuous.

Proposition 3.15 *Assume (3.3), (3.4), (3.8), and (3.14). Then for every $R_0 > 0$ and every*

$$P \in \mathcal{L}(X^\alpha, X^\beta) \quad \textit{with} \quad \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \le R_0$$

and for every $\gamma, \gamma' \in I$ such that

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \ge \gamma, \tag{3.19}$$

there exist constants $\omega = \omega(\gamma', R_0) \ge 0$ and $M_0 = M_0(\gamma, \gamma', R_0)$ such that, for $t > 0$,

$$\|S_P(t)u_0\|_{\gamma'} \le M_0 e^{\omega t} t^{-(\gamma' - \gamma)} \|u_0\|_\gamma, \quad \gamma' \ge \gamma. \tag{3.20}$$

In particular $S_P(t) \in \mathcal{L}(X^\gamma)$ and it is a semigroup of linear continuous operators in X^γ.

Proof. First, by (3.14) and (3.19), see (3.16), we can use part iii) in Lemma 3.9 for the fixed point of $\mathcal{F}$, with $\gamma' = \alpha$, $0 \le \varepsilon = \delta = \alpha - \gamma < 1$, to get

$$|||u(\cdot; u_0)|||_{\alpha,\varepsilon} \le |||S(\cdot)u_0|||_{\alpha,\varepsilon} + C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u(\cdot; u_0)|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T)T^{\beta+1-\alpha}$.

Then, note that, by (3.5) and the choice of ε, $|||S(\cdot)u_0|||_{\alpha,\varepsilon} \le M_0(\gamma, \alpha, T)\|u_0\|_\gamma$ and by (3.8), take T such that $C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \le \frac{1}{2}$ for all perturbations P as in the statement. Thus,

$$|||u(\cdot; u_0)|||_{\alpha,\varepsilon} \le 2M_0(\gamma, \alpha, T)\|u_0\|_\gamma. \tag{3.21}$$

Now by (3.19), we can use part iii) in Lemma 3.9 for the fixed point of $\mathcal{F}$, with $\gamma' \ge \gamma$, $\delta = \gamma' - \gamma$, $0 \le \varepsilon = \alpha - \gamma < 1$, to get

$$|||u(\cdot; u_0)|||_{\gamma',\delta} \le |||S(\cdot)u_0|||_{\gamma',\delta} + C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u(\cdot; u_0)|||_{\alpha,\varepsilon}.$$

again with $C(T) = M_1(T)T^{\beta+1-\alpha}$.

Then, note that, by (3.5) and the choice of δ, $|||S(\cdot)u_0|||_{\gamma',\delta} \le M_0(\gamma, \gamma', T)\|u_0\|_\gamma$ and using (3.21), we have

$$|||u(\cdot; u_0)|||_{\gamma',\delta} \le \Big(M_0(\gamma, \gamma', T) + C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} 2M_0(\gamma, \alpha, T)\Big)\|u_0\|_\gamma.$$

Hence, by the choice of T above,

$$|||u(\cdot; u_0)|||_{\gamma',\delta} \le \tilde{M}_0(\gamma, \gamma', T)\|u_0\|_\gamma$$

with $\tilde{M}_0(\gamma, \gamma', T) = M_0(\gamma, \gamma', T) + M_0(\gamma, \alpha, T)$.

Note that this gives,

$$\|S_P(t)\|_{\gamma,\gamma'} \le \frac{\tilde{M}_0(\gamma, \gamma', T)}{t^{\gamma' - \gamma}}, \qquad \text{for all} \quad 0 < t \le T. \tag{3.22}$$

Arguing as in i) in Lemma 3.2 we conclude (3.20).

In particular, since $X^{\gamma'} \subset X^\gamma$, from (3.20) we get that $S_P(t) \in \mathcal{L}(X^\gamma)$ and is a semigroup of linear continuous operators in X^γ. ∎

Remark 3.16 *Observe that if the original constants $M_0(\beta,\alpha)$ in (3.4), do not depend, or can be taken independent of $\alpha,\beta \in I$, then the same is true for $M_0(\gamma,\gamma',R_0)$ and $\omega(\gamma',R_0)$ in Proposition 3.15, which become independent of the spaces of the scale.*

In any case, once the perturbation P is fixed, the estimate (3.22) for $0 < t \leq 1$ allows to apply part iii) in Lemma 3.2 to obtain that there exists $\omega_0 = \omega_0(P)$ such that

$$\|S_P(t)u_0\|_\gamma \leq M_0(\gamma)\mathrm{e}^{\omega_0 t}\|u_0\|_\gamma$$

for all $\gamma \in E(\alpha) = (\alpha-1,\alpha]$. In turn, part iv) in Lemma 3.2 implies that (3.20) holds for some exponent independent of γ,γ'.

Remark 3.17 Strong solutions

i) Note that for $u_0 \in X^\alpha$ the solution of (3.9) obtained in Proposition 3.12 satisfies

$$u(t;u_0) = S_P(t)u_0 = \mathcal{F}(u,u_0)(t) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau;u_0)\,d\tau.$$

Then, by Lemma 3.10, u is locally Hölder with values in X^α and then $h(\tau) = Pu(\tau)$ is locally Hölder with values in X^γ for any $\gamma \leq \beta$. Since $S(t)$ is analytic, then Lemma 3.2.1 in [6] implies that, for $t > 0$, $u(t;u_0)$ is a C^1 strong solution of

$$u_t + Au = Pu, \qquad \text{in} \quad X^\gamma,$$

where $-A$ is the infinitesimal generator of the semigroup $S(t)$ in X^γ. In particular $-(A-P)$ is the infinitesimal generator of the semigroup $S_P(t)$.

For $u_0 \in X^\gamma$, for $\gamma \in E(\alpha)$, the solution of (3.9) obtained in Theorem 3.13 satisfies $u(t_0) \in X^\alpha$, for any $t_0 > 0$ and we can use the argument above for $t > t_0$ as well.

ii) Assume we can prove that the semigroup $S_P(t)$ is analytic in X^γ. Then, thanks to Proposition 3.15, we can use the Transfer of Analiticity lemma, proved in [4]

Lemma 3.18 **Transfer of Analiticity**

Assume $\{S(t)\}_{t\geq 0}$ is an analytic semigroup in a Banach space X. Assume that for some Banach space Y and for $t > 0$,

$$S(t) \in \mathcal{L}(X,Y).$$

Then for each $u_0 \in X$, the curve of the semigroup $(0,\infty) \ni t \mapsto S(t)u_0$ is analytic in Y. Moreover for each t_0, the Taylor series in Y has a radius of convergence not smaller than the one in X.

In particular if $Y \subset X$, with continuous injection, then $\{S(t)\}_{t\geq 0}$ defines an analytic semigroup in Y. ■

to conclude that $S_P(t)$ defines an analytic semigroup in $X^{\gamma'}$ for $\gamma' \geq \gamma$.

Now we consider the case in which several perturbations are considered sequentially. Assume

$$P^1, P^2 \in \mathcal{L}(X^\alpha, X^\beta), \quad \text{with} \quad 0 \leq \alpha - \beta < 1$$

and consider the semigroup $S_{P^1}(t)$ for $u_0 \in X^\gamma$ with $\gamma \in E(\alpha)$. Now we repeat the construction starting out of $S_{P^1}(t)$. Then we would have the new semigroup that we denote $S_{[P^1,P^2]}(t)$ which is formally given by

$$S_{[P^1,P^2]}(t)u_0 = S_{P^1}(t)u_0 + \int_0^t S_{P^1}(t-\tau)P^2 S_{[P^1,P^2]}(\tau)u_0\, d\tau. \tag{3.23}$$

Now we state some properties of the resulting semigroups.

Lemma 3.19 *i) If $P = aI$, with $a \in I\!R$, then*

$$S_{aI}(t) = \mathrm{e}^{at}S(t) \quad \text{in } X^\gamma \text{ for every } \gamma \in I.$$

ii) If $P \in \mathcal{L}(X^\alpha, X^\beta)$, $0 \leq \alpha - \beta < 1$, and $a \in I\!R$ then

$$S_{[aI,P]}(t) = S_{[P,aI]}(t) = S_{P+aI}(t) = \mathrm{e}^{at}S_P(t) \quad \text{in } X^\gamma \text{ for every } \gamma \in E(\alpha).$$

iii) If $P^1, P^2 \in \mathcal{L}(X^\alpha, X^\beta)$, $0 \leq \alpha - \beta < 1$, then

$$S_{[P^1,P^2]}(t) = S_{[P^2,P^1]}(t) = S_{P^1+P^2}(t) \quad \text{in } X^\gamma \text{ for every } \gamma \in E(\alpha).$$

Proof.
i) Note that for $P = aI$ we can take $\alpha = \beta = \gamma$ for any $\gamma \in I$. Now for $u_0 \in X^\gamma$ we have that $u(t; u_0) = S_{aI}(t)u_0$ is the unique fixed point of (3.9), that is

$$u(t; u_0) = S(t)u_0 + a\int_0^t S(t-\tau)u(\tau; u_0)\, d\tau.$$

On the other hand, setting $v(t) = \mathrm{e}^{at}S(t)u_0$ we have

$$S(t)u_0 + a\int_0^t S(t-\tau)v(\tau)\, d\tau = \Big(1 + \int_0^t a\mathrm{e}^{a\tau}\, d\tau\Big)S(t)u_0 = \mathrm{e}^{at}S(t)u_0 = v(t).$$

Hence, $v(t) = u(t; u_0)$.
ii) From i), applied to $S_P(t)$, we have, for every $u_0 \in X^\gamma$ with $\gamma \in E(\alpha)$, $S_{[P,aI]}(t)u_0 = \mathrm{e}^{at}S_P(t)u_0$ which, by (3.9), can be written as

$$\mathrm{e}^{at}S_P(t)u_0 = \mathrm{e}^{at}S(t)u_0 + \int_0^t \mathrm{e}^{a(t-\tau)}S(t-\tau)P\Big(\mathrm{e}^{a\tau}S_P(\tau)u_0\Big)\, d\tau.$$

On the other hand, by the expression for $S_{aI}(t)$ from i), we have that for every $u_0 \in X^\gamma$ with $\gamma \in E(\alpha)$,

$$S_{[aI,P]}(t)u_0 = \mathrm{e}^{at}S(t)u_0 + \int_0^t \mathrm{e}^{a(t-\tau)}S(t-\tau)PS_{[aI,P]}(\tau)u_0\, d\tau.$$

The uniqueness of the fixed point problem gives

$$S_{[aI,P]}(t)u_0 = \mathrm{e}^{at} S_P(t)u_0 = S_{[P,aI]}(t)u_0.$$

iii) Note that from Remark 3.17, for every $\gamma \in E(\alpha)$ and $u_0 \in X^\gamma$, $u(t; u_0) := S_{P^1+P^2}(t)u_0$ satisfies, for $t > 0$,

$$u_t + Au = (P^1 + P^2)u, \qquad \text{in} \quad X^\gamma,$$

which can be written as

$$u_t + (A - P^1)u = P^2 u,$$

or as

$$u_t + (A - P^2)u = P^1 u.$$

In the first case we get

$$u(t; u_0) = S_{P^1}(t)u_0 + \int_0^t S_{P^1}(t - \tau)P^2 u(\tau; u_0)\, d\tau$$

while in the second

$$u(t; u_0) = S_{P^2}(t)u_0 + \int_0^t S_{P^2}(t - \tau)P^1 u(\tau; u_0)\, d\tau.$$

The uniqueness of solutions of (3.9) implies then that $u(t; u_0) = S_{[P^1,P^2]}(t)u_0$, in the first case and $u(t; u_0) = S_{[P^2,P^1]}(t)u_0$ in the second. ■

From the results above, we obtain in particular, with the notations in Section 2

Theorem 3.20 *Assume that m lies in a bounded set in $L^p(\Omega)$, with $p > N/2$, m_0 lies in a bounded set in $L^r(\Gamma)$ and also that the family of potentials V_ε is a L^r–concentrated bounded family, for $r > N - 1$, that is*

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C, \quad r > N - 1.$$

Then, for any $1 < q < \infty$, the problem (1.3) defines a strongly continuous, order preserving, analytic semigroup, $S_{m,m_0,\varepsilon}(t)$ in the space $H^{2\gamma,q}_{bc}(\Omega)$ for any

$$\gamma \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'}).$$

Moreover the semigroup satisfies the smoothing estimates

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{H^{2\gamma',q}_{bc}(\Omega)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega) \tag{3.24}$$

for every $\gamma, \gamma' \in I(q)$, with $\gamma' \geq \gamma$, for some $M_{\gamma',\gamma}$ and $\mu \in I\!R$ independent of m, m_0 and $0 < \varepsilon \leq \varepsilon_0$ and $\gamma, \gamma' \in I(q)$.

Finally, we have the variations of constants formula

$$S_{m,m_0,\varepsilon}(t)u_0 = u(t;u_0) = S_0(t)u_0 + \int_0^t S_0(t-\tau)S_\varepsilon u(\tau;u_0)\,d\tau, \qquad t>0, \tag{3.25}$$

where S_ε is given in (2.9) and the semigroup $S_{m,m_0,\varepsilon}(t)$ coincides with the one in Corollary 2.6 and gives, for $t>0$, weak solutions of (1.3) as in (2.14).

In particular, if $q>N-1$, then there exists $\gamma' \in I(q)$ such that $H^{2\gamma',q}_{bc}(\Omega)) \subset C^\beta(\overline{\Omega})$ for some $\beta>0$ and the solutions of (1.3) become $C^\beta(\overline{\Omega})$ smooth.

Proof. Note that from Section 2 we have (3.1) and (3.2) so we can apply the results of this section here.

Also, recall that from Lemma 2.1, since $p>N$ and $r>N-1$ we can take $s+\sigma<2$ in (2.5) and (2.6) and then, for any $1<q<\infty$, P_ε, Q_0, R_0 are well defined from $H^{s,q}(\Omega)$ into $H^{-\sigma,q}(\Omega)$ provided $s>1/q$, $\sigma>1/q'$ and

$$2 > s+\sigma > \max\{\frac{N}{p}, 1+\frac{N-1}{r}\} := K. \tag{3.26}$$

Hence $S_\varepsilon := P_\varepsilon + Q_0 + R_0 \in \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))$ and using the embeddings discussed in Section 2, we get

$$S_\varepsilon := P_\varepsilon + Q_0 + R_0 \in \mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega))$$

is continuous and uniformly bounded in norm, for $0<\varepsilon\le\varepsilon_0$, for s,σ in that range. Therefore we can apply Theorem 3.13 and Proposition 3.15 with $\alpha=\frac{s}{2}$ and $\beta=\frac{-\sigma}{2}$ and we get the above results for indexes

$$\gamma \in (\frac{s}{2}-1, \frac{s}{2}], \qquad \gamma' \in [-\frac{\sigma}{2}, 1-\frac{\sigma}{2}), \qquad \gamma' \ge \gamma.$$

Now note that as s,σ range over the set defined by (3.26), then the intervals for γ and γ' above fill the interval $I(q)$.

Also, Proposition 3.15 gives (3.24) and the fact that μ is independent of γ,γ' follows from Lemma 3.2.

Note that the variation of constants formula (3.25) is given by construction. Then, for γ such that $-\frac{1}{2q'} \ge \gamma \ge -1+\frac{1}{2q}$, from Remark 3.7, we have that the semigroup $S_{m,m_0,\varepsilon}(t)$ coincides with the one constructed in Corollary 2.6. In particular it is order preserving. Also, since the latter semigroup is analytic we can use Remark 3.17 to conclude that $S_{m,m_0,\varepsilon}(t)$ is an analytic semigroup as in the statement. Also, we get that the semigroup gives weak solutions of (1.3) as in (2.14). ■

Remark 3.21 *The order preserving property can also be obtained as a consequence of Theorem 4.4 below.*

Then we have the following

Definition 3.22 *If $V_0 \in L^r(\Gamma)$, with $r>N-1$, the semigroup $S_{m,m_0+V_0}(t)$ is defined as in Theorem 3.20 with the choice $V_\varepsilon=0$ and m_0+V_0 replacing m_0.*

Note that for every $u_0 \in H^{2\gamma,q}_{bc}(\Omega)$, with $\gamma\in I(q)$, as in the Theorem, the function $u(t;u_0) := S_{m,m_0+V_0}(t)u_0$ is, for $t>0$, a weak solution of (1.4) in the sense of (2.15).

4 Convergence of linear semigroups

Our goal here is to prove, in particular, that the semigroup $S_{m,m_0,\varepsilon}(t)$ in Theorem 3.20 converges, in a suitable sense, to the semigroup $S_{m,m_0+V_0}(t)$ in Definition 3.22, provided

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon \to V_0, \quad cc - L^r \quad \text{for some } r > N-1.$$

This result will come out of a more general result on the dependence of perturbations.

With the setting of Section 3, assume that we have two perturbations

$$P^i \in \mathcal{L}(X^\alpha, X^\beta), \quad i = 1,2, \quad 0 \le \alpha - \beta < 1.$$

Our goal is then to compare semigroups $S_{P^i}(t)$, $i = 1,2$. Hence assume

$$\|P^i\|_{\mathcal{L}(X^\alpha,X^\beta)} \le R_0 \quad i = 1,2$$

for some $R_0 > 0$. Also, consider the existence and regularity intervals as in (3.19)

$$\gamma \in E(\alpha) = (\alpha-1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta+1), \qquad \gamma' \ge \gamma.$$

Consider then two initial data $u_0^i \in X^\gamma$, $i = 1,2$ and the corresponding solution of (3.9)

$$u^i(t; u_0^i) = S_{P^i}(t)u_0^i = S(t)u_0^i + \int_0^t S(t-\tau)P^i u^i(\tau; u_0^i)\, d\tau, \qquad t > 0$$

and denote

$$z(t, u_0^1, u_0^2) = u^1(t; u_0^1) - u^2(t; u_0^2).$$

Theorem 4.1 *With the notations above, for any $R_0 > 0$,*
i) There exists a sufficiently small T_0 such that for all perturbations P^i such that $\|P^i\|_{\mathcal{L}(X^\alpha,X^\beta)} \le R_0$,

$$|||z(\cdot, u_0^1, u_0^2)|||_{\gamma',\delta} \le L(T_0, R_0)\Big(\|u_0^1 - u_0^2\|_\gamma + \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_\gamma\Big), \tag{4.1}$$

with $\delta = \gamma' - \gamma$. In particular,

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \le \frac{L(T_0,R_0)}{t^{\gamma'-\gamma}}\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \textit{for all} \quad 0 < t \le T_0 \tag{4.2}$$

ii) For every $T > T_0$

$$\|z(t, u_0^1, u_0^2)\|_{\gamma'} \le M_2(T, T_0, R_0)\Big(\|u_0^1 - u_0^2\|_\gamma + \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_\gamma\Big), \quad T_0 \le t \le T. \tag{4.3}$$

In particular,

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \le L(T, T_0, R_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \textit{for all} \quad T_0 < t \le T \tag{4.4}$$

Proof.
i) We first show the estimate for short times. Dropping momentarily the dependence in u_0^1, u_0^2, we get

$$z(t) = S(t)(u_0^1 - u_0^2) + \int_0^t S(t-\tau)\Big(P^1 - P^2\Big)u^2(\tau)\, d\tau + \int_0^t S(t-\tau)P^1 z(\tau)\, d\tau.$$

First note that by (3.20) in Proposition 3.15 we have, for $\varepsilon = \alpha - \gamma$ and for any $T > 0$,

$$|||u^i|||_{\alpha,\varepsilon} \leq M_0(T, R_0)\|u_0^i\|_\gamma. \tag{4.5}$$

Then, arguing as in Lemma 3.9, we get, with $\delta = \gamma' - \gamma$,

$$|||z|||_{\gamma',\delta} \leq |||S(\cdot)(u_0^1 - u_0^2)|||_{\gamma',\delta} + C(T)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}|||u^2|||_{\alpha,\varepsilon} + C(T)\|P^1\|_{\mathcal{L}(X^\alpha,X^\beta)}|||z|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T)T^{\beta+1-\delta}$. Also note that the first term in the right hand side is bounded by $M(T)\|u_0^1 - u_0^2\|_\gamma$.

First, with $\gamma' = \alpha$, $\delta = \alpha - \gamma = \varepsilon$, we get

$$|||z|||_{\alpha,\varepsilon} \leq M(T)\|u_0^1 - u_0^2\|_\gamma + C(T)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}|||u^2|||_{\alpha,\varepsilon} + C(T)\|P^1\|_{\mathcal{L}(X^\alpha,X^\beta)}|||z|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T)T^{1+\beta-\alpha}$. Then for T_0 small such that $C(T_0)R_0 \leq 1/2$ we get

$$|||z|||_{\alpha,\varepsilon} \leq 2M(T_0)\|u_0^1 - u_0^2\|_\gamma + 2C(T_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}|||u^2|||_{\alpha,\varepsilon}. \tag{4.6}$$

Now with γ' and $\delta = \gamma' - \gamma$ and $\varepsilon = \alpha - \gamma$, we get

$$|||z|||_{\gamma',\delta} \leq M(T_0)\|u_0^1 - u_0^2\|_\gamma + C(T_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}|||u^2|||_{\alpha,\varepsilon} + C(T_0)\|P^1\|_{\mathcal{L}(X^\alpha,X^\beta)}|||z|||_{\alpha,\varepsilon}$$

again with $C(T_0) = M_1(T_0)T_0^{1+\beta-\alpha}$.

Hence, using (4.5) and (4.6), we get (4.1) which is valid for all P^i such that $\|P^i\|_{\mathcal{L}(X^\alpha,X^\beta)} \leq R_0$.

In particular, if $u_0^1 = u_0^2 = u_0$ then

$$|||z|||_{\gamma',\delta} \leq L(T_0, R_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0\|_\gamma$$

which leads to (4.2).
ii) For $T_0 < t \leq T$ observe that

$$u^i(t; u_0^i) = S_{P^i}(t)u_0^i = S(t - T_0)u^i(T_0; u_0^i) + \int_{T_0}^t S(t-\tau)P^i u^i(\tau; u_0^i)\, \mathrm{d}\tau.$$

Dropping momentarily the dependence in u_0^1, u_0^2, we get

$$z(t) = S(t - T_0)z(T_0) + \int_{T_0}^t S(t-\tau)\Big(P^1 - P^2\Big)u^2(\tau)\, d\tau + \int_{T_0}^t S(t-\tau)P^1 z(\tau)\, d\tau.$$

and then

$$\|z(t)\|_{\gamma'} \leq M(T)\|z(T_0)\|_{\gamma'} + K(T)\|P^1 - P^2\|_{\alpha,\beta} \int_{T_0}^{t} (t-\tau)^{-(\gamma'-\beta)} \|u^2(\tau)\|_{\alpha}\, d\tau +$$

$$+K(T)\|P^1\|_{\alpha,\beta} \int_{T_0}^{t} (t-\tau)^{-(\gamma'-\beta)} \|z(\tau)\|_{\alpha}\, d\tau.$$

Now, by (4.5), u^2 is bounded in X^α on $[T_0, T]$ and then the second term above is bounded by

$$K_1(T) \left(\int_{T_0}^{t} (t-\tau)^{-(\gamma'-\beta)}\, d\tau \right) \|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)} \sup_{[T_0,T]} \|u^2(t)\|_{\alpha}$$

which, using (4.5), is bounded by

$$K_2(T, T_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)} \|u_0^2\|_{\gamma}.$$

So we end up with

$$\|z(t)\|_{\gamma'} \leq M(T)\|z(T_0)\|_{\gamma'} + K_2\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_{\gamma} + K_2\|P^1\|_{\alpha,\beta} \int_{T_0}^{t} (t-\tau)^{-(\gamma'-\beta)} \|z(\tau)\|_{\alpha}\, d\tau$$

for all $T_0 \leq t \leq T$.

Then using the singular Gronwall lemma, see Lemma 7.1.1, page 188, [6], we conclude

$$\|z(t)\|_{\gamma'} \leq M_2(T)\Big(\|z(T_0)\|_{\gamma'} + \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_{\gamma}\Big), \quad T_0 \leq t \leq T.$$

Using now the estimate for short times, (4.1), we get (4.3). In particular, if $u_0^1 = u_0^2 = u_0$ then we get (4.4). ■

Remark 4.2 *Observe that if both semigroups $S_{P^1}(t)$ and $S_{P^2}(t)$ decay exponentially, we actually get*

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \leq \frac{L(R_0)e^{-\omega t}}{t^{\gamma'-\gamma}} \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \textit{for all} \quad 0 < t < \infty$$

for some $\omega > 0$.

In the general case, if we replace P^1 and P^2 by $P^1 - \lambda I$ and $P^2 - \lambda I$ such that both $S_{P^1-\lambda I}(t)$ and $S_{P^2-\lambda I}(t)$ decay exponentially we get

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \leq \frac{L(R_0)e^{\omega t}}{t^{\gamma'-\gamma}} \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \textit{for all} \quad 0 < t < \infty$$

for some $\omega \in \mathbb{R}$.

From the Theorem we get the following

Corollary 4.3 *Given $P^1 \in \mathcal{L}(X^\alpha, X^\beta)$, assume for some $\gamma \in E(\alpha) = (\alpha-1, \alpha]$ we have*

$$\|S_{P^1}(t)\|_{\gamma,\gamma} \le M e^{\omega_0 t}, \quad \textit{for all} \quad t > 0$$

for some $M = M(\gamma)$ and $\omega_0 \in I\!\!R$.

Then for any $\varepsilon > 0$, if $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ is sufficiently small we have for any $\gamma' \in E(\alpha) = (\alpha-1, \alpha]$

$$\|S_{P^2}(t)\|_{\gamma',\gamma'} \le M' e^{(\omega_0+\varepsilon)t}, \quad \textit{for all} \quad t > 0$$

for some M' depending on $M, \omega_0, \gamma'\varepsilon$.

In particular, $S_{P^1}(t)$ decays exponentially, that is if $\omega_0 < 0$, then so does $S_{P^2}(t)$ if $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ is sufficiently small.

Finally $S_{P^1}(t)$ and $S_{P^2}(t)$ satisfy the estimates (3.20) with $\omega = \omega_0 + \varepsilon$.

Proof. First observe that from part iii) in Lemma 3.2 we have that the exponential bounds for $S_{P^1}(t)$ and $S_{P^2}(t)$ are independent of γ; see (3.6) and (3.7). Therefore it is enough to prove the result for the given $\gamma \in E(\alpha) = (\alpha-1, \alpha]$.

Now for $\varepsilon > 0$ note that $e^{-(\omega_0+\varepsilon)t} S_{P^1}(t) = S_{P^1-(\omega_0+\varepsilon)I}(t)$ decays exponentially in X^γ. In particular there exists t_0 such that $\delta := \|S_{P^1-(\omega_0+\varepsilon)I}(t_0)\|_{\gamma,\gamma} < 1$. Then, from Theorem 4.1, if $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ is sufficiently small we have $\delta' := \|S_{P^2-(\omega_0+\varepsilon)I}(t_0)\|_{\gamma,\gamma} < 1$. Then the last part of part iii) in Lemma 3.2 implies that $e^{-(\omega_0+\varepsilon)t} S_{P^2}(t) = S_{P^2-(\omega_0+\varepsilon)I}(t)$ decays exponentially in X^γ too and the result follows.

The estimates (3.20) with $\omega = \omega_0 + \varepsilon$ follows from part iv) in Lemma 3.2. ■

With this we can finally prove

Theorem 4.4 *Assume*

$$m_\varepsilon \to m \quad \textit{in } L^p(\Omega), \quad p > \frac{N}{2},$$

$$m_{0,\varepsilon} \to m_0 \quad \textit{in } L^r(\Gamma), \quad r > N-1,$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \textit{for some } r > N-1$$

and for any $1 < q < \infty$, consider the semigroups $S_{m,m_0,\varepsilon}(t)$ and $S_{m,m_0+V_0}(t)$ obtained in Theorem 3.20 and in Definition 3.22.

Then for every

$$\gamma, \gamma' \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'}) \quad \gamma' \ge \gamma,$$

and $T > 0$ we have that there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t) - S_{m,m_0+V_0}(t)\|_{\mathcal{L}(H^{2\gamma,q}_{bc}(\Omega), H^{2\gamma',q}_{bc}(\Omega))} \le \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \textit{for all} \quad 0 < t \le T.$$

In particular, if $q > N-1$, then there exists $\gamma' \in I(q)$ such that $H^{2\gamma',q}_{bc}(\Omega)) \subset C^\beta(\overline{\Omega})$ for some $\beta > 0$ and the solutions of (1.3) converge to solutions of (1.4) uniformly in $\overline{\Omega}$.

Proof. Note that is was proved in Lemma 2.5 in [5] that

$$P_\varepsilon \to P_0 \quad \text{in} \quad \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))$$

where P_ε is defined in (2.3) and P_0 in (2.11). In the setting here this translates into

$$P_\varepsilon \to P_0 \quad \text{in} \quad \mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega)).$$

Then the rest follows from Theorem 4.1. ■

Remark 4.5 *In particular we can obtain again that the semigroups $S_{m,m_0,\varepsilon}(t)$ and $S_{m,m_0+V_0}(t)$ are order preserving. For this, note that taking C^1 smooth m_ε, $m_{0,\varepsilon}$ and V_0, the results in [2] imply that the semigroups are order preserving. Then the convergence above shows the same property for the limiting semigroups.*

Remark 4.6 *Note that after Theorems 3.20 and 4.4 we can consider problems of the type (1.3) and (1.4) only assuming C^1 regularity on the diffusion coefficient $a(x)$.*

Finally, about the optimal exponential bound for the semigroups $S_{m,m_0,\varepsilon}(t)$ we have the following

Proposition 4.7 *Assume*

$$m_\varepsilon \to m \quad \textit{in } L^p(\Omega), \quad p > \frac{N}{2},$$

$$m_{0,\varepsilon} \to m_0 \quad \textit{in } L^r(\Gamma), \quad r > N-1,$$

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon \to V_0, \quad cc - L^r \quad \textit{for some } r > N-1$$

and denote by λ_1^ε the first eigenvalue of the following eigenvalue problem

$$\left\{ \begin{array}{rcll} -div(a(x)\nabla\varphi^\varepsilon) + c(x)\varphi^\varepsilon &=& m_\varepsilon(x)\varphi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon} V_\varepsilon(x)\varphi^\varepsilon + \lambda\varphi^\varepsilon & \textit{in } \Omega \\ a(x)\frac{\partial\varphi^\varepsilon}{\partial\vec{n}} + b(x)\varphi^\varepsilon &=& m_{0,\varepsilon}(x)\varphi^\varepsilon & \textit{on } \Gamma \\ \mathcal{B}\varphi^\varepsilon &=& 0 & \textit{on } \partial\Omega\setminus\Gamma. \end{array} \right.$$

i) We have that

$$\lambda_1^\varepsilon \to \lambda_1^0$$

which is the first eigenvalue of the limit eigenvalue problem

$$\left\{ \begin{array}{rcll} -div(a(x)\nabla\varphi) + c(x)\varphi &=& m(x)\varphi + \lambda\varphi & \textit{in } \Omega, \\ a(x)\frac{\partial\varphi}{\partial\vec{n}} + b(x)\varphi &=& (m_0(x) + V_0(x))\varphi & \textit{on } \Gamma, \\ \mathcal{B}\varphi &=& 0 & \textit{on } \partial\Omega\setminus\Gamma. \end{array} \right.$$

ii) For sufficiently small ε and for any $-\mu < \lambda_1^0$ the semigroups $S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)$ and $S_{m,m_0+V_0}(t)$ obtained in Theorem 3.20 and in Definition 3.22 satisfy that for any $1 < q < \infty$ and for every

$$\gamma \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$$

$$\|S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)u_0\|_{H^{2\gamma,q}_{bc}(\Omega)} \leq M_\gamma e^{\mu t}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t>0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega)$$

and

$$\|S_{m,m_0+V_0}(t)u_0\|_{H^{2\gamma,q}_{bc}(\Omega)} \leq M_\gamma e^{\mu t}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t>0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega)$$

with M_γ independent of ε. Consequently, there semigroups satisfy (2.13) and (3.24) for such μ and constants independent of ε.

Proof. Note that the eigenvalue problems above are the ones associated with the generators of the semigroups $S_{m_\varepsilon,m_{0,\varepsilon},\varepsilon}(t)$ and $S_{m,m_0+V_0}(t)$, which have only discrete spectrum. Then part i) follows from Corollary 4.2 and Remark 4.3 in [5].

For part ii) note that the exponential bound of the semigroup using a lower bound on the real part of the spectrum of the generator follows from Theorem 1.3.4 in [6]. This combined with part i) and Lemma 3.2, gives the rest. Note that the result also follows from Corollary 4.3. ■

5 Final results and remarks

Note that from Theorems 3.20 and 4.4 using standard techniques for semigroups, [6, 10, 3, 9], as well as the results in [2] it is easy to analyze the solutions of nonhomogeneous problems like

$$\left\{\begin{array}{rcll} u^\varepsilon_t - \mathrm{div}(a(x)\nabla u^\varepsilon) + c(x)u^\varepsilon &=& \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}f_\varepsilon(x,t) + g(x,t) & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon &=& j(x,t) & \text{on } \Gamma \\ \mathcal{B}u^\varepsilon &=& 0 & \text{on } \partial\Omega\setminus\Gamma \\ u^\varepsilon(0) &=& u_0 & \text{in } \Omega \end{array}\right. \tag{5.1}$$

assuming

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon \to V_0, \quad cc-L^r \quad \text{for some } r>N-1 \tag{5.2}$$

and suitable conditions of the type

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}f_\varepsilon(\cdot,t) \to f_0(\cdot,t), \quad cc-L^r \quad \text{for some } r>N-1 \tag{5.3}$$

for each $t>0$. In such a case the limit problem reads

$$\left\{\begin{array}{rcll} u_t - \mathrm{div}(a(x)\nabla u) + c(x)u &=& g(x,t) & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& V_0(x)u + j(x,t) + f_0(x,t) & \text{on } \Gamma \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega\setminus\Gamma \\ u(0) &=& u_0 & \text{in } \Omega. \end{array}\right. \tag{5.4}$$

Themain tool here would be the variations of constants formula. Details are left for the reader.

On the other hand, note that given V_0 and f_0 on Γ we can define V_ε and f_ε extending the functions V_0 and $f_0(\cdot,t)$ to ω_ε in the direction of the normal. That is, if $z \in \omega_\varepsilon$ then z can be written in a unique way as $z = x - \sigma\vec{n}(x)$, for some $\sigma \in (0,\varepsilon)$. Therefore, we define $V_\varepsilon(z) = V_0(x)$ and $f_\varepsilon(z,t) = f_0(x,t)$. With this definition we may easily prove that if $V_0 \in L^\rho(\Gamma)$, $f_0(\cdot,t) \in L^r(\Gamma)$, then (5.2), (5.3) hold.

In particular, in case the domain is not smooth, it may be difficult to give a meaning to the boundary condition in (5.4), although (5.1) has a natural and simple variational formulation not involving surface integrals or traces. Hence the limit functions of (5.1) can be taken as proper way of defining solutions of (5.4) in such a case.

Finally, it is not difficult to see that all previous results can be carried out with minor changes to the case in which the region ω_ε collapses to a regular orientable hyper–surface $\Gamma \subset \overline{\Omega}$, not necessarily the boundary of the domain. In such a case, for the problem

$$\left\{\begin{array}{rcll} u^\varepsilon_t - \mathrm{div}(a(x)\nabla u^\varepsilon) + c(x)u^\varepsilon &=& \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}f_\varepsilon(x,t) + g(x,t) & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon &=& j(x,t) & \text{on } \partial\Omega_R \\ u^\varepsilon &=& 0 & \text{on } \partial\Omega \\ u^\varepsilon(0) &=& u_0 & \text{in } \Omega \end{array}\right.$$

the limit problem reads

$$\left\{\begin{array}{rcll} u_t - \mathrm{div}(a(x)\nabla u) + c(x)u &=& V_0(x)\delta_\Gamma u + f_0(x,t)\delta_\Gamma + g(x,t) & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& j(x,t) & \text{on } \partial\Omega_R \\ u &=& 0 & \text{on } \partial\Omega_D \\ u(0) &=& u_0 & \text{in } \Omega, \end{array}\right.$$

where we denote by $f_0\delta_\Gamma$ and $V_0\delta_\Gamma u$ the functionals $< f_0\delta_\Gamma, \varphi >= \int_\Gamma f_0\varphi$ and $< V_0 u, \varphi >= \int_\Gamma V_0 u\varphi$. Here we also denote by $\partial\Omega_R$ and $\partial\Omega_D$ a partition of $\partial\Omega$ where Robin and Dirichlet type boundary condition are imposed, respectively.

Observe also that by taking test functions with support near points on Γ it is easy to see that the limit problem is in fact a transmission problem across Γ, where the jump condition reads

$$[u]_\Gamma = 0, \quad a(x)[\frac{\partial u}{\partial \vec{n}}]_\Gamma - V_0(x)u = f_0(x,t), \quad x \in \Gamma \quad t > 0.$$

See [5] for a further discussion in the case of elliptic problems.

Last but not least, observe that as mentioned in Section 2 the results in this paper apply to the case when the diffusion coefficient is a positive definite matrix instead of a scalar coefficient. First order coefficients can be handled as well.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, Boston, 1978.

[2] H. Amann, "Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems", in: *Schmeisser/Triebel: Function Spaces, Differential Operators and Nonlinear Analysis*, Teubner Texte zur Mathematik, **133**, 9-126 (1993).

[3] H. Amann, *Linear and Quasilinear Parabolic Problems. Abstract Linear Theory.* Birkäuser Verlag, 1995.

[4] J.M. Arrieta, J.W. Cholewa, T.Dlotko, and A.Rodríguez-Bernal, "Linear parabolic equations in locally uniform spaces". Math. Models Methods Appl. Sci., vol 14, 253–293 (2004).

[5] J.M. Arrieta, A. Jiménez-Casas, A. Rodríguez-Bernal, "Nonhomogeneous flux condition as limit of concentrated reactions ", Revista Iberoamericana de Matematicas vol 24,no 1, 183-211, (2008).

[6] D.Henry, "Geometry Theory of Semilinear Parabolic Equations", Springer-Verlag, (1981).

[7] A. Jiménez-Casas, A. Rodríguez-Bernal "Asymptotic behaviour of a parabolic problem with terms concentrated in the boundary", Nonlinear Analysis, Theory Methods and Applications, 71, 2377-2383 (2009).

[8] T. Kato, "Perturbation theory for linear operators". Grundlehren der Mathematischen Wissenschaften, 132. Springer-Verlag, Berlin-New York, (1976).

[9] A.Lunardi, "Analytic semigroups and optimal regularity in parabolic problems", Birkhauser (1995).

[10] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer (1983).

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